Algèbre de Lie:

le 18.09.19

Rem: Sophus Lie (Norway ~1880):

Lie groups → Lie algebras

Basic notions:

Déf: An algebra (over a field $\mathbb{F}$) is a vector space A equiped with a multiplication : $\cdot : A \times A \to A$ which is a bilinear

(A bilin: $(\alpha x)\cdot y = x(\alpha y) = \alpha(xy) \quad \forall \alpha \in \mathbb{F},\ x, y \in A$

$(x+y)\cdot z = xz + yz,\ x(y+z) = xy + xz \quad \forall x, y, z \in A$)

Déf: a) an algebra A is commutative if $xy = yx \quad \forall x, y \in A$

b) an algebra A is associative if $(xy)z = x(yz) \quad \forall x, y, z \in A$

Ex: $A = \mathbb{F}$ (eg: $A = \mathbb{C}$)

Ex: $A = \mathrm{Mat}(n, \mathbb{C})$, $(A\cdot B)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$ is not commut. but associative.

Ex: $A = \mathrm{Mat}(n, \mathbb{C})$, $A * B = AB + BA$ (obv. bilinear)

it is commutative.

$(A * B) * C = (AB + BA) * C = ABC + CAB + CBA + BAC$

$A * (B * C) = ABC + BCA + ACB + CBA$ non-associative

Déf: A lie algebra (over a field $\mathbb{F}$) is an algebra over $\mathbb{F}$ with a multiplication $[\cdot,\cdot]$ such that:

i) $[x, y] = -[y, x] \quad \forall x, y \in A$

(the Jacobi identity)

ii) $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0 \quad \forall x, y, z \in A$

Rem: a LA (lie algebra) is not commutative (in general)

$$\big([[x,y],z] - [x,[y,z]] = [[x,y],z] + [[y,z],x] = -[[z,x],y]\big)$$

and is not associative (in general)

Thm 1: If char $\mathbb{F} \neq 2$, then 1) $\Leftrightarrow$ 1') $[x,x] = 0 \quad \forall x \in A$

Proof: $\Leftarrow$: $0 = [x+y, x+y] = \underset{\underset{0}{\|}}{[x,x]} + \underset{\underset{0}{\|}}{[y,y]} + [x,y] + [y,x]$

so 1') $\Rightarrow$ 1)

$\Rightarrow$: take $y = x$, $[x,x] = -[x,x] \Rightarrow 2[x,x] = 0 \Rightarrow [x,x] = 0$

(if char $\mathbb{F} \neq 2$) ▨

Ex: (very trivial case)

A is any vector space with $[x,y] = 0$

Thm 2: (very important case)

Let A be an associative algebra (with $\cdot$). Let A^- be the same vector space with $[x,y] := xy - yx$. Then A^- is a LA.

Proof: $[\cdot,\cdot]$ is a bilinear operation:

$$[\alpha x, y] = \alpha xy - \alpha yx = \alpha[x,y]$$

$$[x+y, z] = \ldots = xz + yz - zx - zy = [x,z] + [y,z]$$

$$[y,x] = yx - xy = -(xy - yx) = -[x,y]$$

$$[[x_1,x_2],x_3] = [x_1x_2 - x_2x_1, x_3] = \underset{1\;2\;3}{x_1x_2x_3} - \underset{2\;1\;3}{x_2x_1x_3}$$

$$- \underset{3\;1\;2}{x_3x_1x_2} + \underset{3\;2\;1}{x_3x_2x_1}$$

$$[[x_2,x_3],x_1] = \ldots = 231 - 132 - 123 + 213$$

$$[[x_3,x_1],x_2] = \ldots = 312 - 321 - 231 + 132$$

and the sum of them $= 0$. Jacobi identity holds for $[.,.]$ ▨

Algèbre de Lie:

?

le 18.09.19

Ex: $A = \mathrm{Mat}(n, \mathbb{C})$, then $[x, y] = xy - yx$ is the commutator
them
of x and y.

$A^- = gl(n, \mathbb{C})$ - the general linear algebra

Def: Let A be a L.A. A subset $\mathcal{E} \subset A$ is a basis of A if every $x \in A$ is uniquely represented as $x = \sum_i \alpha_i e_i$, $e_i \in \mathcal{E}$

$\mathrm{card}(\mathcal{E}) = \dim A$.

Ex: $A = gl(2, \mathbb{C})$ $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\mathcal{E} = \{e_{11}, e_{12}, e_{21}, e_{22}\}$ is a basis of A.

$\dim(gl(2, \mathbb{C})) = 4$.

Ex: $A = gl(n, \mathbb{C})$ $\dim(gl(n, \mathbb{C})) = n^2$ $e_{ij} = \begin{array}{c} \quad j \\ \left(\begin{array}{c|c|c} 0 & & 0 \\ \hline & 1 & \\ \hline 0 & & 0 \end{array}\right) i \end{array}$

$$[e_{ij}, e_{km}] = e_{i\underbrace{j\,e_{k}}m} - e_{k\underbrace{m\,e_{i}}j} = \delta_{jk}\, e_{im} - \delta_{mi}\, e_{kj}$$

Def: Let A be a LA. Let $e_1 \dots e_n$ be a bassis of A then

$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^{k} e_k$ are the defining relations of A.

$\{c_{ij}^{k}\}$ are the structure constents of A.

Rem: if $x, y \in A$ then $x = \sum_{i=1}^{n} \alpha_i e_i$ and $y = \sum_{j=1}^{n} \beta_j e_j$

$$[x, y] = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j [e_i, e_j] = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_i \beta_j c_{ij}^{k} e_k$$

Subalgebras:

Def: Let A be a LA. A vector subspace $B \subset A$ is a Lie subalgebra of A if $[x, y] \in B$ $\forall x, y \in B$ (ie: B is closed in respect to $[\cdot, \cdot]$)

Ex: (Trivial)

A is a Lie subalgebra of A $\{0\}$ is a Lie subalgebra of A.

Ex: if $x \in A$, then $B = \text{span}(x) = \{ y \in A \mid y = \alpha \cdot x \,,\ \alpha \in \mathbb{F} \}$

B is a Lie subalgebra of A, because:

$y_1, y_2 \in B$ then $y_1 = \alpha_1 x$ and $y_2 = \alpha_2 x$ so $[y_1, y_2] = \alpha_1 \alpha_2 [x, x] = 0 \in B$

Ex: $A = gl(n, \mathbb{C})$, $b^+ = \{ x \in gl(n, \mathbb{C}) \mid x_{ij} = 0 \text{ if } i > j \}$ 

b^+ is a Lie subalgebra.

Rem: $\text{Tr}\, X = \sum_{i=1}^{n} X_{ii}$, $\text{Tr}(XY) = \sum_{i=1}^{n} (XY)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij} Y_{ji} = \sum_{j=1}^{n} \sum_{i=1}^{n} Y_{ji} X_{ij} = \text{Tr}(YX)$

Def: $\{ x \in gl(n, \mathbb{C}) \mid \text{Tr}\, x = 0 \} =: \underline{sl(n, \mathbb{C})}$ – the special linear algebra.

Rem: $sl(n, \mathbb{C})$ is a Lie subalgebra of $gl(n, \mathbb{C})$

a) if $x, y \in sl(n, \mathbb{C})$ then $\text{Tr}(\alpha x + \beta y) = \alpha \text{Tr}\, x + \beta \text{Tr}\, y = 0$

$\Rightarrow sl(n, \mathbb{C})$ is a vector subspace of $gl(n, \mathbb{C})$

b) if $x, y \in sl(n, \mathbb{C})$ then $\text{Tr}\,[x, y] = \text{Tr}(xy - yx)$

$= \text{Tr}(xy) - \text{Tr}(yx) = 0 \Rightarrow sl(n, \mathbb{C})$ is a Lie subalgebra.

Ex: $sl(2, \mathbb{C})$, <u>the canonical basis</u>: $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$[e, f] = h \,,\quad [h, e] = 2e \,,\quad [h, f] = -2f.$

Ex: Does $sl(2, \mathbb{C})$ have a 2-dim Subalgebra?

Yes! $B = \text{span}(h, e)$ or $B = \text{span}(h, f)$

because $[h, e] = 2e$ and $[h, f] = -2f$.

Ex: $sl(n, \mathbb{C})$ a basis: e_{ij}, $i \neq j$, $\text{Dim}\, sl(n, \mathbb{C}) = n^2 - n$

and $h_i = e_{ii} - e_{i+1, i+1}$, $i = 1 \ldots n-1$ $\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 \ddots \end{pmatrix}, \ldots \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}$

le 25.09.19

Algèbre de lie :

The center of a LA

⚠ jusque là tout SL(n, ℂ) est sL(n, ℂ) ⚠

Def: Let A be a LA, then $Z(A) = \{x \in A \mid [x,y]=0 \ \forall y \in A\}$ is the center

Thm 3: $Z(A)$ is a subalgebra of A.

Proof: Let $x_1, x_2 \in Z(A)$, then $[\alpha_1 x_1 + \alpha_2 x_2, y] = \alpha_1 [x_1, y] + \alpha_2 [x_2, y] = 0$

$\Rightarrow Z(A)$ is a vector subspace.

$[x_1, x_2] = 0 \in Z(A)$

▨

Thm 4: a) $Z(gL(n,\mathbb{C})) = \mathrm{Span}\{E\} = \{x = \alpha E \mid \alpha \in \mathbb{C}\}$ E is the unit matrix

b) $Z(sL(n,\mathbb{C})) = \{0\}$

Proof: a) if $x \in Z(gL(n,\mathbb{C}))$ then $[x, e_{ii}] = 0 \Leftrightarrow e_{ii}x = xe_{ii}$

$\Rightarrow e_{ii} x e_{jj} = x e_{ii} e_{jj} = 0$

$\Rightarrow x_{ij} = 0$ if $i \neq j$ $\Rightarrow$ x is a diagonal matrix

$x = \sum_{i=1}^{n} \alpha_i e_{ii}$, $[x, e_{\ell m}] = 0$

$\Rightarrow [x, e_{\ell m}] = \sum_{i=1}^{n} \alpha_i [e_{ii}, e_{\ell m}] = \sum_{i=1}^{n} \alpha_i (e_{ii} e_{\ell m} - e_{\ell m} e_{ii})$

$= \sum_{i=1}^{n} \alpha_i (\delta_{i\ell} e_{im} - \delta_{mi} e_{i\ell}) = \alpha_\ell e_{\ell m} - \alpha_m e_{\ell m} = 0 \Rightarrow \alpha_\ell = \alpha_m \ \forall \ell, m$

$\Rightarrow x = \alpha \sum_{i=1}^{n} e_{ii} = \alpha E$

b) let $x \in Z(sL(n,\mathbb{C})) \Rightarrow x \in Z(gL(n,\mathbb{C}))$ $\rightarrow \mathrm{Tr}\,y' = 0$

┌ because if $y \in gL(n,\mathbb{C})$, then $y = \overbrace{(y - \frac{1}{n}\mathrm{Tr}\,y\ E)}^{y'} + \frac{1}{n}\mathrm{tr}\,y\ E$

$[x,y] = [x,y'] + [x, \frac{1}{n}\mathrm{tr}\,y\ E] = [x,y'] + 0 = 0$

$\Rightarrow Z(SL(n,\mathbb{C})) \subset Z(gL(n,\mathbb{C}))$ ┘

$\Rightarrow Z(sL(n,\mathbb{C})) = 0$

▨

Def: A L.A is abelian if $Z(A) = A$

Ex: Any 1-dimentional L.A.

Ex: $B = \{ x \in gl(n,\mathbb{C}) \mid x_{ij} = 0 \text{ if } i \neq j \}$

Thm 5: Let A be a 2-dim. L.A. then either A is abelian or there exists a basis of A $\mathcal{E} = \{e_1, e_2\}$ s.t. $[e_1, e_2] = e_2$

Proof: Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of A then $[e_1, e_2] = \alpha e_1 + \beta e_2$.

$$\Rightarrow \begin{cases} \text{either } \alpha = \beta = 0 \\ \text{or either } \alpha \neq 0 \text{ or } \beta \neq 0 \text{ , suppose } \beta \neq 0. \end{cases}$$

Take $e_1' = \frac{1}{\beta} e_1$, $e_2' = \alpha e_1 + \beta e_2$ then $\mathcal{E}' = \{e_1', e_2'\}$ is a basis

$[e_1', e_2'] = e_2'$

☒

Rem: If A is 2-dim , $[e_1, e_2] = e_2$

$x = \alpha e_1 + \beta e_2$, if $\in Z(A)$, $[e_1, x] = \beta [e_1, e_2] = \beta e_2$, $[x, e_2] = \alpha e_2$

then $Z(A) = \{0\}$

From Lie groups to Lie algebras:

Def: A group is a set G with an operation $\cdot : G \times G \rightarrow G$ such that:

1) $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3) \quad \forall g_1, g_2, g_3 \in G$
2) $\exists\, e \in G$ s.t $e \cdot g = g \cdot e = g \quad \forall g \in G$
3) if $g \in G$, $\exists\, g^{-1} \in G$ s.t. $g g^{-1} = g^{-1} g = e$

Def: A subgroup of a group G is a subset $H \subset G$ s.t. $g_1 \cdot g_2 \in H \;\forall g_1, g_2 \in H$.

Ex: $GL(n,\mathbb{C}) = \{ g \in Mat(n,\mathbb{C}) \mid \det g \neq 0 \}$ the general linear group

$GL^+(n,\mathbb{C}) = \{ g \in Mat(n,\mathbb{C}) \mid \det g > 0 \}$ subgroup of $GL(n,\mathbb{C})$

$\{ g \in Mat(n,\mathbb{C}) \mid \det g < 0 \}$ not a group

$SL(n,\mathbb{C}) = \{ g \in Mat(n,\mathbb{C}) \mid \det g = 1 \}$ group (special lin. grp)

$SL(n,\mathbb{C}) < GL^+(n,\mathbb{C}) < GL(n,\mathbb{C})$.

le 25.09.19

Algèbre de Lie:

Rem: The function $\|.\|_\infty : \mathrm{Mat}(n,\mathbb{C}) \longrightarrow \mathbb{R}$ given by $\|g\|_\infty = \max_{i,j} |g_{i,j}|$ is a norm on $\mathrm{Mat}(n,\mathbb{C})$

Déf: Let $\{g_n\}_{n=1}^{\infty} \subset \mathrm{Mat}(n,\mathbb{C})$ then $g_n \xrightarrow{n\to\infty} g$ if $\|g_n - g\| \xrightarrow{n\to\infty} 0$

Déf: A matrix Lie group G is a subgroup of $GL(n,\mathbb{C})$ such that if $\{g_n\}_{n=1}^{\infty} \subset G$ and $g_n \xrightarrow{n\to\infty} g$ then either $g \in G$ or $\det g = 0$.

Ex: $SL(n,\mathbb{C})$ is a Lmatrix Lie group.

(if $\det g_n = 1$ and $g_n \underset{n\to\infty}{\longrightarrow} g$ then $\det g = 1$ because det is a continuous fonction)

Ex: $GL^+(n,\mathbb{C})$ is a matrix Lie group

(if $\det g_n > 0$ and $g_n \underset{n\to\infty}{\longrightarrow} g$ then $\det g > 0$ or $\det g = 0$)

Ex: $G = \{g \in GL(n,\mathbb{R}) \mid g_{ij} \in \mathbb{Q}\}$ it's a group but not a matrix lie group.

Lem: Let $x \in \mathrm{Mat}(n,\mathbb{C})$ then $e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}$ is a convergent serie

Proof: $(\mathrm{Mat}(n,\mathbb{C}), \|.\|_\infty)$ is a Banach space

let $A, B \in \mathrm{Mat}(n,\mathbb{C})$, then $\|A \cdot B\|_\infty \leq n \|A\|_\infty \|B\|_\infty$

$\Rightarrow \|x^k\| \leq n^{k-1} \|x\|^k$, let $g_m = \sum_{k=0}^{m} \frac{x^k}{k!}$ $\quad t = n\|x\|$

$$\|g_m - g_s\| = \|\sum_{k=s+1}^{m} \frac{x^k}{k!}\| \leq \frac{1}{n} \sum_{k=s+1}^{m} \frac{(n \|x\|)^k}{k!} = \frac{1}{n} \sum_{k=s+1}^{m} \frac{t^k}{k!} \xrightarrow[s\to\infty]{} 0$$

$\Rightarrow \{g_m\}_{m=1}^{\infty}$ is a Cauchy sequence

▨

Lem: Let $x \in \mathrm{Mat}(n,\mathbb{C})$, $\det(e^x) = e^{\mathrm{tr}\, x}$

Proof: $\exists S \in \mathrm{Mat}(n,\mathbb{C})$ s.t $x = S \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} S^{-1}$ and then $x^k = S \left[\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}^k \right] S^{-1}$

$= S \begin{pmatrix} \lambda_1^k & & * \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix} S^{-1}$, $e^x = S e^{\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}} S^{-1} = S \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} \lambda_1^k & & * \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix} S^{-1}$

$= S \begin{pmatrix} e^{\lambda_1} & & * \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} S^{-1}$, $\det(x) = \det \begin{pmatrix} e^{\lambda_1} & & * \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} = e^{\lambda_1} \cdots e^{\lambda_n}$

$= e^{\sum \lambda_i} = e^{\mathrm{Tr}\, x}$ ■

Rem: $\lim_{m\to\infty} (1+\frac{a}{m})^m = e^a$

Thm 6: Let $x,y \in \mathrm{Mat}(n,\mathbb{C})$, then

a) $\left(e^{\frac{1}{m}x} \cdot e^{\frac{1}{m}y}\right)^m \xrightarrow[m\longrightarrow\infty]{} e^{x+y}$ [Lie product formula]

$$e^{1/m\,x} = E + \frac{1}{m}x + \dots \qquad e^{1/m\,x}\cdot e^{1/m\,y} = E + \frac{1}{m}(x+y) + \dots$$

$$e^{1/m\,y} = E + \frac{1}{m}y + \dots$$

not a proof just remark of how it works

b) $e^x \cdot e^y = e^{x+y+\frac{1}{2}[x,y]+\frac{1}{12}[x,[x,y]]-\frac{1}{12}[y,[x,y]]+\dots\dots}$

commutator of higher order [the Bauer-Campbell Hausdorff formula] [B.C.H]

$[x,y] = xy - yx$

Rem: if $[x,[x,y]] = [y,[x,y]] = 0$ then $e^x e^y = e^{x+y+\frac{1}{2}[x,y]} = e^{x+y} e^{\frac{1}{2}[x,y]}$

Déf: Let $G < GL(n,\mathbb{C})$ be a matrix Lie group, then $\mathfrak{g}(G) = \{x \in \mathrm{Mat}(n,\mathbb{C}) \mid e^{tx} \in G, \forall t \in \mathbb{R}$

is a Lie algebra of the Lie group G.

---

le 02.10.19

Thm 4: $\mathfrak{g}(G)$ equipped with the map $[.,.] : \mathfrak{g}(G)\times\mathfrak{g}(G) \to \mathfrak{g}(G)$ is a Lie algebra over $\mathbb{R}$. $\hookrightarrow [x,y] = xy - yx$

Proof: a) if $x \in \mathfrak{g}(G)$ and $s \in \mathbb{R}$ then $e^{t(s\cdot x)} = e^{(t\cdot s)x} \in G$

if $x,y \in \mathfrak{g}(G)$, $e^{t/m\,x}, e^{t/m\,y} \in G$ $\forall t \in \mathbb{R}, m \in \mathbb{N}$

$\Rightarrow g_m = (e^{t/m\,x} e^{t/m\,y})^m \in G$

$e^{t(x+y)} = \lim_{m\to\infty} g_m \in G$ G is a matrix Lie group

$\Rightarrow x+y \in \mathfrak{g}(G)$ $\det(e^{t(x+y)}) = e^{t\,tr(x+y)} \neq 0$

b) $[x,y] = xy - yx$ satisfies the Jacoby identity.

but we have to show that $[x,y] \in \mathfrak{g}(G)$

Lemma: if $x \in \mathfrak{g}(G)$ and $g \in G$ then $gxg^{-1} \in \mathfrak{g}(G)$

Pf: $e^{t(gxg^{-1})} = \sum_{n\geq 0} \frac{t^n}{n!}(gxg^{-1})^n = g e^{tx} g^{-1} \in G$ ∎

le 02.10.19

Algèbre de Lie:

Proof: (Suite)

b) let $x, y \in \mathcal{G}(G)$, $\frac{1}{t}(e^{tx} y e^{-tx} - y) \in \mathcal{G}(G)$ $\forall t \in \mathbb{R}$

$\uparrow$ finite dim. vector space is closed.

$$\lim_{t \to 0} \frac{1}{t}(e^{tx} y e^{-tx} - y) \in \mathcal{G}(G)$$

$$= \lim_{t \to 0} \frac{1}{t}\left[(E + tx + \dots) y (E - tx - \dots) - y\right] =$$

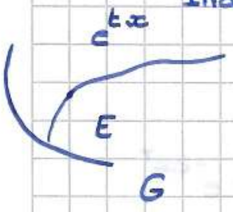
$$= \lim_{t \to 0} \frac{1}{t}\left[y + txy - tyx \cdot y + o(t)\right] = [x, y]$$

$$\Rightarrow [x, y] \in \mathcal{G}(G)$$

$\blacksquare$

Rem: if $x \in \mathcal{G}(G)$, then $G_x = \{g \in G \mid g = e^{tx}, t \in \mathbb{R}\}$ is a one-parameter subgroup of G.

Indeed: $g_1 \cdot g_2 = e^{t_1 x} \cdot e^{t_2 x} = e^{(t_1 + t_2) x}$



if $g = e^{tx}$ then $g^{-1} = e^{-tx}$ $\quad (e^{tx} \cdot e^{-tx} = e^{tx - tx} = E)$

and $E = e^{0x}$

Ex: 1) $G = \left\{ g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a, b \in \mathbb{C}, a \neq 0 \right\}$ is a group

$g_1 \cdot g_2 = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \in G$, $\quad g^{-1} = \begin{pmatrix} 1/a & -b/a \\ 0 & 1 \end{pmatrix} \in G$

G is a matrix Lie group.

$x \in \mathcal{G}(G) \Leftrightarrow e^{tx} = \begin{pmatrix} a(t) & b(t) \\ 0 & 1 \end{pmatrix}$ $\quad \forall t \in \mathbb{R}$

if $t \simeq 0$ $\quad e^{tx} = E + tx + \dots = \begin{pmatrix} 1 + tx_{11} & tx_{12} \\ tx_{21} & 1 + tx_{22} \end{pmatrix} \Rightarrow \begin{cases} x_{21} = 0 \\ x_{22} = 0 \end{cases}$

we have a necessary condition for $x \in \mathcal{G}(G)$.

it's also a sufficient condition:

$$e^{t\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}} = E + \sum_{k \geq 1} \frac{t^k}{k!} \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}^k = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \in G$$

Thus $\mathcal{G}(G) = \{ x \in gL(2, \mathbb{C}) \mid x_{21} = x_{22} = 0 \}$

2) $G = GL(n, \mathbb{C})$

$x \in \mathfrak{g}(G) \Leftrightarrow e^{tx} \in GL(n,\mathbb{C}) \Leftrightarrow \det(e^{tx}) = e^{t\, tr x} \neq 0$

thus $\mathfrak{g}(G) = gl(n, \mathbb{C})$.

3) $G = GL^{+}(n, \mathbb{C})$

$x \in \mathfrak{g}(G) \Leftrightarrow e^{tx} \in G \Leftrightarrow \det(e^{tx}) > 0 \Leftrightarrow e^{t \cdot tr x} > 0$

$\Leftrightarrow tr x \in \mathbb{R}$

Thus $\mathfrak{g}(G) = \{x \in gl(n,\mathbb{C}) \, / \, tr x \in \mathbb{R}\}$ is a vector space over $\mathbb{R}$

4) $G = SL(n, \mathbb{C})$

$x \in \mathfrak{g}(G) \Leftrightarrow e^{tx} \in G \Leftrightarrow \det e^{tx} = 1 \Leftrightarrow e^{t\, tr x} = 1$

$\Leftrightarrow tr x = 0$

Thus $\mathfrak{g}(G) = sl(n, \mathbb{C})$ (by definition)

5) $G = O(n) := \{ g \in GL(n, \mathbb{R}) \mid g^T \cdot g = E \}$ - the orthogonal group

(g^T is the transposed matrix)

$x \in \mathfrak{g}(G) \Leftrightarrow e^{tx} \in G \Leftrightarrow e^{tx^T} \cdot e^{tx} = E \; \forall t \in \mathbb{R} \Leftrightarrow e^{tx^T} = e^{-tx^T}$

$\Rightarrow$ sufficient condition $x^T = -x$

[Rem: $(e^x)^T = \left(\sum_{k \geq 0} \frac{x^k}{k!} \right)^T = \sum_{k \geq 0} \frac{(x^k)^T}{k!} = e^{x^T}$]

it is also a necessary condition:

$t \approx 0 : \; E + tx^T + ... = E - tx + ... \Rightarrow x^T = -x$

Thus, $\mathfrak{g}(G) = \{ x \in gl(n, \mathbb{R}) \mid x^T + x = 0 \}$ anti-sym matrix

Rem: if x, y are anti-sym. matrices, then

$([x,y])^T = (xy - yx)^T = y^T x^T - x^T y^T = (-y)(-x) - (-x)(-y)$

$= -[x,y] \Rightarrow [x,y]$ is anti-sym.

le 02.10.19

Algèbre de Lie:

6) $SO(n) = \{ g \in SL(n, \mathbb{R}) \mid g^T \cdot g = E \}$ - the special ortho. group.

$$x \in \mathcal{G}(SO(n)) \Leftrightarrow e^{t \cdot x} \in O(n) \text{ and } \det(e^{tx}) = 1$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$x^T + x = 0 \qquad\qquad tr x = 0$$

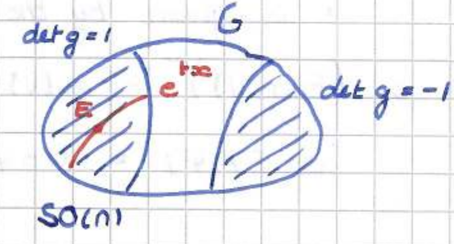
$$\hookrightarrow Tr(x^T + x) = 2 tr x = 0$$

thus $\mathcal{G}(SO(n)) = so(n) = o(n)$

Rem: $g^T g = E \Rightarrow \det g = \pm 1$

$\Rightarrow O(n)$ is not connected

connexe



the unitary group.

4) $\mathcal{U}(n) := \{ g \in GL(n, \mathbb{C}) \mid g^* g = E \}$ $\quad (g^* = \bar{g}^t = \overline{g^t}) \; ((e^x)^* = e^{x^*}$

$$x \in \mathcal{G}(\mathcal{U}(n)) \Leftrightarrow e^{tx} \in \mathcal{U}(n) \Leftrightarrow (e^{tx})^* e^{tx} = E \quad \forall t \in \mathbb{R}$$

$$\Leftrightarrow e^{tx^*} = e^{-tx}$$

a sufficient cond. : $x^* = -x$

it's also a necessary condition:

$$t \approx 0 : \quad E + tx^* + \ldots = E - tx - \ldots \quad \Rightarrow x^* = -x$$

Thus $u(n) = \{ x \in gl(n, \mathbb{C}) \mid x^* + x = 0 \}$ anti-Hermitian matrix.

is a real LA. (if $x^* = -x$, then $(ix)^* = -ix^* = ix$)

8) $S\mathcal{U}(n) = \{ g \in SL(n, \mathbb{C}) \mid g^* \cdot g = E \}$ the special unitary group

$$x \in \mathcal{G}(S\mathcal{U}(n)) \Leftrightarrow e^{tx} \in S\mathcal{U}(n) \Leftrightarrow x^* + x = 0 \text{ and } tr x = 0$$

Thus $su(n) = \{ x \in sl(n, \mathbb{C}) \mid x^* + x = 0 \}$

Rem: $su(n) \subset u(n)$ as a Lie subalgebra.

le 09.10.12

Derivation:

Déf: Let V be a vector space (over a Field $\mathbb{F}$) then End(V) is the set of all linear maps from V to V.

Rem: a) End(V) is a vector space (if we set $(\alpha\varphi + \beta\psi)v = \alpha\varphi(v) + \beta\psi(v)$)

b) End(V) is an associative algebra if the multiplication $\cdot$ is given by the composition of maps $\psi \cdot \varphi = \psi \circ \varphi$

c) $(\mathrm{End}(V))^- = gl(V)$ is a lie algebra (if the Lie bracket is $[\varphi, \psi] = \varphi \circ \psi - \psi \circ \varphi$)

Déf: Let A be an algebra.

a) $D \in gl(A)$ is a derivation if $D(x \cdot y) = D(x) \cdot y + x D(y)$ $\forall x, y \in A$

b) [a particular case] if A is a LA then $D \in gl(A)$ is a derivation if $D([x, y]) = [Dx, y] + [x, Dy]$ $\forall x, y \in A$.

c) Der(A) is the set of all derivations of A.

Rem: $D \in \mathrm{Der}(A)$ is named a derivation by analogy with the Leibniz rule: $(f \cdot g)' = f' \cdot g + f \cdot g'$ for functions.

Thm 8: Let A be an algebra (or a LA). Then Der(A) is a Liesubalgebra of $gl(A)$.

Proof: i) Der(A) is a vector subspace

$$(\alpha \cdot D_1 + \beta D_2)(x \cdot y) = \alpha D_1(x \cdot y) + \beta D_2(x \cdot y) =$$

$$\alpha [D_1(x) y + x D_1(y)] + \beta [D_2(x) y + x D_2(y)]$$

$$= (\alpha D_1(x) + \beta D_2(x))\, y + x(\alpha D_1(y) + \beta D_2(y))$$

ii) $$[D_1, D_2](xy) = (D_1 \circ D_2 - D_2 \circ D_1)(xy)$$

$$= D_1(D_2(xy)) - D_2(D_1(xy)) = D_1(D_2(x)y + xD_2(y)) - D_2(D_1(x)y + xD_1(y)$$

$$= D_1(D_2(x))y + D_2(x)D_1(y) + D_1(x)D_2(y) + xD_1(D_2(y)) - \ldots$$

$$= [D_1, D_2](x)\, y + x\, [D_1, D_2](y) \quad \forall x, y \in A \Rightarrow [D_1, D_2] \in \mathrm{Der}(A) \quad ∎$$

le 09.10.12

Algèbre de Lie:

Rem: Der(A) is a LA.

Rem: Der(Der(A)) is a LA......

Ex: Let A be an abelian LA.

$Der(A) = gl(A)$

if $D \in gl(A)$ $\quad \underbrace{D([x,y])}_{0} = \underbrace{[D(x),y]}_{0} + \underbrace{[x,D(y)]}_{0}$

Ex: 2-dim A, $[e_1,e_2] = e_2$

$D(e_1) = a_{11}e_1 + a_{12}e_2$ and $D(e_2) = a_{21}e_1 + a_{22}e_2$

$D([e_1,e_2]) = [D(e_1),e_2] + [e_1, D(e_2)] = a_{11}e_2 + a_{22}e_2$

$D(\overset{\shortparallel}{e_2}) = a_{11}e_2 + a_{22}e_2 = a_{21}e_1 + a_{22}e_2$

$\Rightarrow a_{11} = a_{21} = 0$

$\Rightarrow Dim(Der(A)) = 2$

A basis of Der(A): $\quad D_1(e_1) = 0, \quad D_1(e_2) = e_2$

$\mathcal{E} = \{D_1, D_2\}$: $\quad D_2(e_1) = e_2, \quad D_2(e_2) = 0$

Rem: $D_1(y) = [e_1, y] \quad \forall y \in A$

$D_2(y) = [-e_2, y] \quad \forall y \in A$

Déf: Let A be a LA and $x \in A$. Then $ad_x \in gl(A)$ (the adjoint map) is the linear map s.t. $ad_x y = [x,y] \quad \forall y \in A$.

Thm 9: $ad_x \in Der(A)$

Proof: $ad_x([y,z]) = [x,[y,z]] = -[[y,z],x] = [[x,y],z] + [[z,x],y]$

$= [[x,y],z] + [y,[x,z]] = [ad_x(y),z] + [y, ad_x(z)]$

$\forall x,y \in A$ $\qquad \blacksquare$

Def: Let A be a LA then $D \in Der(A)$ is an inner derivation if $\exists x \in A$ s.t. $D(y) = [x,y]$ $\forall y \in A$ $(D = ad_x)$

Otherwise, D is an outer derivation.

Ex: A, $\dim A = 2$, $[e_1, e_2] = e_2$ all derivations are inner

Ex: if A is abelian, D is inner only if $D = 0$

Ideals : Très important

Déf: Let A be a LA and $I \subset A$ be a vector subspace. I is an ideal of A if $[x,y] \in I$ $\forall x \in A$, $y \in I$.

I is a characteristic ideal of A if $D(y) \in I$ $\forall D \in Der(A)$, $y \in I$

Rem: Any characteristic ideal is an ideal (since $ad_x \in Der(A)$)

Any ideal is a subalgebra of A.

Ex:

A	I	subalgebra	ideal	char. ideal
A	A	✓	✓	✓
A	$\{0\}$	✓	✓	✓
A	$Z(A)$	✓	✓	✓ (*)
$A, [e_1,e_2]=e_2$	$Span\{e_1\}$	✓	×	×
	$Span\{e_2\}$	✓	✓	✓
$sl(2,\mathbb{C})$	$Span(h, e)$	✓	×	×

(*) $y \in Z(A)$, $D \in Der(A)$

$$[x, D(y)] = D([x,y]) - [D(x), y] = 0 \Rightarrow D(y) \in Z(A)$$

Def: Let A be a LA, let $I, J \subset A$ be vector subspaces.

$[I,J]$ = vector space generated by all $[x,y]$, $x \in I$, $y \in J$

$(= Span([x,y] \mid x \in I, y \in J))$

le 09.10.19

Algèbre de lie :

Rem: $[I, J] = [J, I]$

thm 10: If I and J are ideals of A / or charatcristic ideals of A

a) then $[I, J]$ is an ideal / characteristic ideal.

b) then $I \cap J$ is an ideal / characteristic ideal.

Proof: a) if $x \in [I, J]$ then $x = \sum_i [y_i, z_i]$ $y_i \in I$, $z_i \in J$

$$[\underset{A}{\overset{\cap}{w}}, x] = \mathrm{ad}_w(x) = \sum_i \mathrm{ad}_w([y_i, z_i]) = \sum_i ([\mathrm{ad}_w y_i, z_i] + [y_i, \mathrm{ad}_w z_i])$$

$\mathrm{ad}_w(y_i) = [w, y_i] \in I$ and $\mathrm{ad}_w(z_i) = [w, z_i] \in J$.

$\Rightarrow [w, x] \in [I, J]$

$\Rightarrow [I, J]$ is an ideal

if I and J are char. ideals and $x \in [I, J]$

$$x = \sum_i [y_i, z_i] \ , \ D(x) = \sum_i ([\underset{I}{\overset{\cap}{D(y_i)}}, z_i] + [y_i, \underset{J}{\overset{\cap}{D(z_i)}}]) \in [I, J]$$

$\blacksquare$

b) exercise.

Rem: if I, J are ideals then $[I, J] \subset I \cap J$

if I, J are subalgebras :

$A = SL(2, \mathbb{C})$, $I = \mathrm{Span}(e)$, $J = \mathrm{span}(f)$, $J \cap I = \{0\}$

$[I, J] = \mathrm{span}(h)$

Déf: Let A be a LA , then $A' = [A, A]$ is the derived algebra of A.

Rem: Other notation : $A' = A^{(1)} = \mathcal{D}A$

Rem: A' is a charact. ideal (by previous thm)

Déf: a LA A is perfect if $A' = A$.

Ex: $A = sl(2, \mathbb{C})$ $[e, f] = h$, $[h, f] - 2f$, $[h, e] = 2e$

$sl(2, \mathbb{C})$ is perfect.

Ex: A - 2 dim $[e_1, e_2] = e_2$, $A' = \mathrm{Span}(e_2)$

Exc: $A' = \{0\} \Leftrightarrow$ A is abelian

le 16.10.19

Déf: The derived series of a LA A is.
$A^{(1)} = A' = [A,A]$, $A^{(2)} = [A', A']$, ... $A^{(n+1)} = [A^{(n)}, A^{(n)}]$

Rem: $A^{(k)}$ is an ideal of A (by thm 10)

Rem: $A \supset A^{(1)} \supset A^{(2)} \supset \dots A^{(k)} \supset A^{(k+1)}$ $\left(A^{(k+1)} = [A^{(k)}, A^{(k)}] \subset A^{(k)} \cap A^{(k)} = A^{(k)} \right)$

Déf: A LA A is solvable if $\exists k \geqslant 1$ s.t. $A^{(k)} = \{0\}$
what does it implies for the abelianity of A, $A^{(k-1)}$?

Déf: The lower central series of a LA is $A_{(1)} = A' = [A,A]$,
$A_{(2)} = [A, A']$ ... $A_{(k+1)} = [A, A_{(k)}]$

Rem: $A_{(k)}$ is an ideal of A (thm 10)

Rem: $A \supset A_{(1)} \supset A_{(2)} \dots$ $\left(A_{(k+1)} = [A, A_{(k)}] \subset A \cap A_{(k)} = A_{(k)} \right)$

Déf: A LA A is nilpotent if $\exists n \geqslant 1$ st. $A_{(k)} = \{0\}$

Thm 11: Every nilpotent LA is solvable.

Proof: $A^{(1)} = A' = A_{(1)}$, then by induction,
$A^{(k)} \subset A_{(k)}$:

$$A^{(k+1)} = [\underset{A}{\underset{\cap}{A^{(k)}}}, \underset{A_{(k)}}{\underset{\cap}{A^{(k)}}}] \subset [A, A_{(k)}] = A_{(k+1)}$$

Hence if $A_{(k)} = \{0\} \Rightarrow A^{(k)} = \{0\}$

■

Exc:

A	nilp.	solv
A is abelian	✓	✓
$n^+(3, \mathbb{C})$	✓	✓

$A' = \{0\} = A_{(1)}$

$\begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$ $[e_{12}, e_{23}] = e_{13}$
$[e_{12}, e_{13}] = [e_{23}, e_{13}] = 0$

$A' = \text{Span}(e_{13})$

$A^{(2)} = \{0\} = A_{(2)}$

le 16.10.19

Algèbre de Lie :

Exc : (Suite)

A	nilp.	solv	
$[e_1,e_2]=e_2$	×	✓	$A'=\text{Span}\{e_2\}=A_{(1)}$ $A^{(2)}=\{0\}$ $A_{(2)}=\text{Span}\{e_2\}=A_{(1)}$
$sL(2,\mathbb{C})$	×	×	$[e,f]=h$, $[f,h]=2f$ $[h,e]=2e$ $A'=A$

Def: A LA A is simple if $\dim A \geqslant 2$ and its only ideals are $\{0\}$ and A.

Rem : Every simple LA is perfect
(A' is an ideal $\Rightarrow$ either $A'=\{0\}$ $\Leftrightarrow$ A is abelian
then any 1-dim subalgebra is an ideal
or $A'=A$.)

Rem: Every solvable LA is not simple
($A' \neq A$ since A is solvable but A' is an ideal
so $A'=\{0\}$ $\Leftrightarrow$ A is abelian)

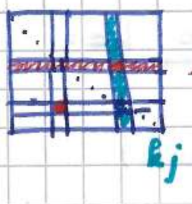
Rem : if $Z(A) \neq \{0\}$ then A is not simple
(because $Z(A)$ is an ideal)

Thm 12 : $A = sL(n,\mathbb{C})$ is simple for any $n \geqslant 2$

Lm : Let $A = sL(n,\mathbb{C})$. Let $I \subset A$ be an ideal. If $e_{ij} \in I$ for some $i \neq j$, then $I = A$.

ij' start

Proof: $I \ni [e_{ij}, e_{jm}] = e_{im}$ for $m \neq i$



$I \ni [e_{ki}, e_{ij}] = e_{kj}$ for $k \neq j$

Repeat for e_{im}, e_{nj} $\Rightarrow$ $\underset{k \neq m}{e_{km}} \in I$ (except possibly for e_{ji})

$$I \ni [\underbrace{[e_{ij}, e_{ji}]}_{\in I}, e_{ji}] = [e_{ii} - e_{jj}, e_{ji}]$$

$$= - e_{ji} - e_{ji} = -2 e_{ji}$$

$\Rightarrow$ all off-diagonal generators are in I

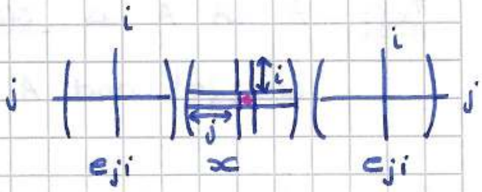
For diagonal elements : a basis : $e_{ii} - e_{i+1\, i+1}$ $\begin{pmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & -1 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}$

$e_{ii} - e_{i+1\, i+1} = [e_{i\, i+1}, e_{i+1\, i}]$

$\Rightarrow I = A$

▨

Proof: (thm)



Let $x \in I$, I is an ideal of A.

Suppose that $x_{ij} \neq 0$ for some $i \neq j$

$$I \ni [\underbrace{[x, e_{ji}]}_{\overset{\cap}{I}}, e_{ji}] = [x e_{ji} - e_{ji} x, e_{ji}] = -2 e_{ji} x e_{ji} = -2 x_{ij} e_{ji}$$

$\Rightarrow e_{ji} \in I$

$\Rightarrow I = A$

lem

if x is diagonal , $x = \sum_{k=1}^{n} \alpha_k e_{km}$, $\sum_{k=1}^{n} \alpha_k = 0$

$$I \ni [e_{ij}, x] = \sum_{k=1}^{n} \alpha_k [e_{ij}, e_{km}] = \sum_{k=1}^{n} \alpha_k (\delta_{jk} e_{ik} - \delta_{ik} e_{kj})$$

$= \alpha_j e_{ij} - \alpha_i e_{ij} = (\alpha_j - \alpha_i) e_{ij} = 0$ if I is a nontrivial ideal

$\Rightarrow \alpha_j = \alpha_i$ $\Rightarrow$ $\alpha_k = \alpha$ $\forall k$ $\Rightarrow (\sum \alpha_k = 0) \Rightarrow \alpha = 0 \Rightarrow x = 0$

▨

le 16.10.19

Algèbre de Lie :

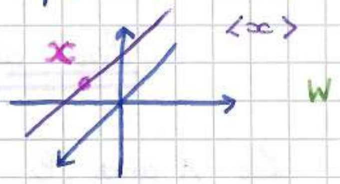
Lie Factor Algebras :

Lemma : Let V be a vector space (over a field $\mathbb{F}$) and let $W \subset V$ be a vector subspace. Then the relation $\overset{W}{\sim}$ defined as : $x \overset{W}{\sim} y \Leftrightarrow x - y \in W$ is a relation of equivalence

$$\ulcorner \; x \overset{W}{\sim} x \; ; \quad x \overset{W}{\sim} y \Rightarrow y \overset{W}{\sim} x \; ; \quad x \overset{W}{\sim} y , \; y \overset{W}{\sim} z \Rightarrow x \overset{W}{\sim} z \; \lrcorner$$

Def : $\langle x \rangle = \{ y \in V \mid y - x \in W \}$ is an equivalence class

Rem : $\langle x \rangle$ is an affine subspace



Ex : $\langle 0 \rangle = W$

Déf : $V/_W = \{ \langle x \rangle \mid x \in V \}$ is the factor space.

Lemma : V/W equipped with operations

$\underset{\mathbb{F}}{\overset{}{\alpha}} \langle x \rangle := \langle \alpha x \rangle$ and $\langle x \rangle + \langle y \rangle := \langle x + y \rangle$

is a vector space

Proof : independence of the choice of representatives

if $\langle x_1 \rangle = \langle x_2 \rangle$ then $\alpha \langle x_1 \rangle = \langle \alpha x_1 \rangle = \langle \alpha x_2 + \overset{W}{\overset{\in}{\alpha (x_1 - x_2)}} \rangle = \langle \alpha x_2 \rangle$

if $\langle x_1 \rangle = \langle x_2 \rangle$, $\langle y_1 \rangle = \langle y_2 \rangle$
$$\langle x_1 \rangle + \langle y_1 \rangle = \langle x_2 + y_2 + \overbrace{(\underbrace{x_1 - x_2}_{\in W} + \underbrace{y_1 - y_2}_{\in W})}^{\in W} \rangle = \langle x_2 + y_2 \rangle = \langle x_2 \rangle + \langle y_2 \rangle$$

■

Thm B : Let A be a LA and let $I \subset A$ be an ideal. Then A/I equipped with a Lie bracket $[\langle x \rangle , \langle y \rangle]_{A/I} = := \langle [x, y]_A \rangle$ is a LA.

Proof: The Lie bracket is defined correctly.

$\langle x_1\rangle = \langle x_2\rangle \quad , \quad \langle y_1\rangle = \langle y_2\rangle$

$[\langle x_1\rangle, \langle y_1\rangle]_{A/I} = \langle [x_1, y_1]_A\rangle = \langle [x_2, y_1]_A + [x_1 - x_2, y_1]_A\rangle$

$= \langle [x_2, y_1]_A\rangle = \langle [x_2, y_2]_A + [x_2, y_1 - y_2]_A\rangle = [\langle x_2\rangle, \langle y_2\rangle]_{A/I}$

$[\,,]_{A/I}$ is bilinear, anti-symmetric.

$[[\langle x_1\rangle, \langle x_2\rangle]_{A/I}, \langle x_3\rangle]_{A/I} + \dots = [\langle [x_1, x_2]_A\rangle, \langle x_3\rangle]_{A/I} + \dots$

$= \langle [[x_1, x_2]_A, x_3]_A\rangle + \dots = \langle [[x_1, x_2]_A, x_3]_A + \dots \rangle = 0$ ▨

---

le 23.10.19

Lie algebra homomorphisms:

Def: Let A and B be LAs. Let f be a linear map between A and B

a) f is a Lie algebra homomorphism if $f([x,y]_A) = [f(x), f(y)]_B$ $\forall x, y \in A$

b) f is a Lie algebra isomorphism if f is bijective and f is a Lie algebra homom.

(thus A and B are isomorphic LA, $A \simeq B$)

c) f is a Lie algebra automorphism if $B = A$ and f is an isomorphism.

Rem: a Lie algebra homomorphism does not have to be injective or surjective.

Rem: if $A \simeq B$, then $\dim A = \dim B$

Ex: a LA A, let I be an ideal of A. Let $\pi : A \to A/I$ be the linear map given by $\pi(x) = \langle x\rangle$.

Then π is a Lie algebra hommomorphism.

$$\underset{[\pi(x), \pi(y)]_{A/I}}{\underset{\parallel}{[\langle x\rangle, \langle y\rangle]_{A/I}}} = \underset{\pi([x,y]_A)}{\underset{\parallel}{\langle [x,y]_A\rangle}}$$

π is surjective but not injective (if $I \neq \{0\}$)

le 23.10.19

Algèbre de Lie :

Ex : A - 2-dim , $[e_1,e_2] = e_2$

B - $sL(2,\mathbb{C})$, $\{e,f,h\}$, $[e,f]=h$, $[h,e]=2e$, $[h,f]=-2f$

$\varphi : A \longrightarrow B$ a linear map such that

$\varphi(e_1) = \frac{1}{2} h$, $\varphi(e_2) = e$

$\varphi([e_1,e_2]) = \varphi(e_2) = e$

$\varphi([e_1,e_2]) = [\varphi(e_1),\varphi(e_2)] = [\frac{1}{2}h, e] = \frac{1}{2}(2e) = e$

it's injective but not surjective.

Ex : $A = sL(2,\mathbb{C})$

Let $\varphi : A \longrightarrow A$ be a linear map such that :

$\varphi(e) = f$, $\varphi(f) = e$, $\varphi(h) = -h$

$\varphi([e,f]) = \varphi(h) = -h$

$[\varphi(e),\varphi(f)] = [f,e] = -[e,f] = -h$

$\varphi([h,e]) = \varphi(2e) = 2f$

$[\varphi(h),\varphi(e)] = [-h,f] = -(-2f) = 2f$

$\varphi \circ \varphi = id_A$ (autom. of order 2)

Def : A Lie algebra automorphisme f is of order $m>1$ if $\underbrace{f\circ f\circ \dots \circ f}_{m} = id$

Ex : $A = gL(n,\mathbb{C})$, let $f : A \longrightarrow A$ be the linear map given by $f(x) = -x^t$. Than f is an automorphism :

$f([x,y]) = -(xy-yx)^t = -y^T x^T + x^t y^t = [y^t, x^t]$

$= [-y^t,-x^t] = [f(y), f(x)]$

Ex : $A = sL(n,\mathbb{C})$, $f : x \longmapsto -x^t$, is a LA automorphism.

if $x \in A \Leftrightarrow tr(x) = 0$ so $tr(-x^t) = 0 \Rightarrow -x^t \in A$

Ex: $\underset{(\nabla)}{A = n^+(3, \mathbb{C})}$, $f: \underset{(\nabla)}{x} \longmapsto \underset{(\Delta)}{-x^t}$

$B = n^-(3, \mathbb{C}) = \{ (\Delta) \}$

then $f: A \to B$ is an isomorphism ($A \simeq_f B$)

Rem: $f \circ f = id$

Ex: $A = gl(n, \mathbb{C})$. Let $M \in Mat(n, \mathbb{C})$ be an invertible matrix

Let f be a linear map : $x \longmapsto MxM^{-1}$

$f([x,y]) = M(xy - yx)M^{-1} = MxyM^{-1} - MyxM^{-1}$

$= MxM^{-1}MyM^{-1} - MyM^{-1}MxM^{-1} = [f(x), f(y)]$

so f is an automorphism.

Ex: $A = sL(n, \mathbb{C})$, $f: x \longmapsto MxM^{-1}$

$x \in A \Leftrightarrow tr(x) = 0$

$tr(f(x)) = tr(MxM^{-1}) = tr(xMM^{-1}) = tr(x) = 0$

$\Rightarrow f(x) \in A$

so f is an automorphism.

Ex: $A = so(n, \mathbb{C})$, $f: x \longmapsto MxM^{-1}$

$x \in A \Leftrightarrow x^t = -x$

$(f(x))^t = (M^{-1})^t x^t M^t = -(M^{-1})^t x M^t \overset{\text{in general}}{\neq} -MxM^t$

Ex: (Lemma)

Let G be a matrix Lie group and $\mathfrak{g}$ be it's Lie algebra.

Let $g \in G$. then the lin. map. $x \longmapsto gxg^{-1}$ is an automorphism.

Proof:

$f([x,y]) = [f(x), f(y)]$

if $x \in \mathfrak{g} \Rightarrow gxg^{-1} \in \mathfrak{g}$ (we had a lemma)

$e^{tgxg^{-1}} = \underbrace{g}_{\in G}\underbrace{e^{tx}}_{\in G}\underbrace{g^{-1}}_{\in G} \in G \Rightarrow f(x) \in \mathfrak{g}$ ∎

Algèbre de Lie:

Thm 14: Let A be a LA, the linear map ad : $A \longrightarrow Der(A)$ given by $ad(x) = ad_x$ $(= [x, \cdot])$ is a Lie algebra homomorphism.

Proof: $ad_x \in Der(A)$ (thm 9) take $z \in A$

$$ad([x,y]_A)(z) = ad_{[x,y]_A}(z) = [[x,y],z] =$$

$$= -[[y,z],x] - [[z,x],y] = [x,[y,z]] - [y,[x,z]]$$

$$= ad_x(ad_y(z)) - ad_y(ad_x(z))$$

$$= [ad_x, ad_y]_{Der(A)}(z) = [ad(x), ad(y)]_{Der(A)}(z) \quad \forall z \in A$$

$$\Rightarrow ad([x,y]_A) = [ad(x), ad(y)]_{Der(A)}$$

■

Rem: $ad_{[x,y]} = [ad_x, ad_y]$

Def: Let A and B be LA. Let $f : A \to B$ be a Lie alg. homomorphism.

$Im\, f = \{y \in B \mid \exists x \in A, \text{ s.t. } y = f(x)\} = Im\, A$ the image of f.

$Ker\, f = \{x \in A \mid f(x) = 0\}$ the kernel of f.

Thm 15:

Let A and B be LA. Let $f : A \to B$ be a Lie alg. homom.

a) If X is a subalgebra of A, then $f(X)$ is a subalgebra of B

b) In particular, $Im\, f$ is a subalgebra of B.

c) If I is an ideal of A and f is surjective then $f(I)$ is an ideal of B.

d) $Ker\, f$ is an ideal of A.

e) If $B = Der(A)$, then $Ker(ad) = Z(A)$

Proof:

a) if $y_1, y_2 \in f(X) \Rightarrow \exists x_1, x_2 \in X$ s.t $f(x_1) = y_1$, $f(x_2) = y_2$

then $[y_1, y_2]_B = [f(x), f(y)]_B = f(\underbrace{[x_1, x_2]_A}_{\in X}) \in f(X)$

b) take $X = A$

c) if $y \in f(I) \Rightarrow x \in I$ s.t $f(x) = y$

take $z \in B$, (f surjective) $\Rightarrow \exists w \in A$ st $f(w) = z$.

$[y, z]_B = [f(x), f(w)]_B = f(\overbrace{[\underset{\substack{\cap \\ I}}{x}, w]_A}^{\in I}) \in f(I) \quad \forall z \in B$

$\Rightarrow f(I)$ is an ideal of B.

d) Let $x \in \operatorname{Ker} f$, take $y \in A$

$f([x, y]_A) = [\underset{\parallel}{f(x)}, f(y)]_B = 0 \Rightarrow [x, y] \in \operatorname{Ker} f \quad \forall y \in A$

$\Rightarrow \operatorname{Ker} f$ is an ideal.

e) $x \in \operatorname{Ker}(ad) \Leftrightarrow ad(x) = ad_x = 0$

$\Leftrightarrow ad_x(y) = 0 = [x, y] \Leftrightarrow x \in Z(A)$ $\boxtimes$

---

le 30.10.19

Thm 16: (Noether)

Let A and B be LA and $f: A \longrightarrow B$ be a lie alg. homomorphism

then $A/\operatorname{Ker} f \simeq \operatorname{Im} f$ ($A/\operatorname{Ker} f$ and $\operatorname{Im} f$ are isomorphic LA)

Proof: By thm 15, $\operatorname{Im} f$ is a subalgebra of B

By thm 16, $\operatorname{Ker} f$ is an ideal of A, so by thm 13

$A/\operatorname{Ker} f$ is a LA.

Define the linear map $\varphi : A/\operatorname{Ker} f \longrightarrow B$ as $\varphi(\langle x \rangle) = f(x)$

if $\langle x \rangle = \langle y \rangle$, $\varphi(\langle x \rangle) = f(x) = f(y + x - y) = f(y) + f(\underbrace{x - y}_{\in \operatorname{Ker} f})$

$= f(y) = \varphi(\langle y \rangle) \Rightarrow$ the definition of φ is correct

le 30.10.19

Algèbre de Lie:

$\text{Ker}\,\varphi = \{ \langle x\rangle \mid \varphi(\langle x\rangle) = 0\} = \{\langle x\rangle \mid f(x) = 0\} = \{\langle x\rangle \mid x \in \text{Ker}\,f\}$

$= \langle 0\rangle \Rightarrow \varphi$ is injective

φ is surjective (as a map onto $\text{Im}\,f$), so

$\varphi : A/\text{Ker}\,f \longrightarrow \text{Im}\,f$ is a bijection (*)

$\varphi([\langle x\rangle, \langle y\rangle]) = \varphi(\langle [x,y]\rangle) = f([x,y])$

$= [f(x), f(y)] = [\varphi(\langle x\rangle), \varphi(\langle y\rangle)]$

$\Rightarrow \varphi$ is a LA homomorphism. (**)

(*)+(**) $\Rightarrow \varphi$ is a LA isomorphism

■

Ex: $A = gl(n,\mathbb{C})$, $f = tr$, $tr : A \longrightarrow \mathbb{C}$

$tr([x,y]_A) = tr(xy - yx) = 0 = [tr(x), tr(y)]_{\mathbb{C}}$

$\Rightarrow tr$ is a LA homomorphisme

$\text{Ker}(tr) = sl(n,\mathbb{C})$ $\quad$ $\text{Im}\,tr = \mathbb{C}$

thm 16

$\Rightarrow gl(n,\mathbb{C}) / sl(n,\mathbb{C}) \; \textcircled{\simeq} \; \mathbb{C}$ it's not clear that it's true.

Rem: if we have $\langle x\rangle \in gl(n,\mathbb{C})/sl(n,\mathbb{C})$ then

$\langle x\rangle = \langle \frac{tr\,x}{n} E\rangle$

because $tr\,x = tr(\frac{tr\,x}{n} E) = \frac{tr\,x}{n} tr\,E = tr\,x$

$\Rightarrow gl(n,\mathbb{C})/sl(n,\mathbb{C}) = \{ t\cdot\langle E\rangle \mid t \in \mathbb{C}\} \simeq \mathbb{C}$

since it's only paramited by $t \in \mathbb{C}$, it's clear

Rem: an "abstract" LA with basis $\mathcal{E} = \{e_1, \dots, e_n\}$ is defined by the relations $[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^{k} e_k$

Thm 17: (Ado - Iwasava)

Let A be a LA (over $\mathbb{F}$) $\dim A < \infty$. Then A is isomorphic to a subalgebra of $gl(n,\mathbb{F})$ for some n.

Ex: A : $[e_1, e_2] = e_2$ $\overset{\text{over } \mathbb{C}}{\vee}$ $\varphi : A \longrightarrow gL(2, \mathbb{C})$

$\varphi(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\varphi(e_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$[\varphi(e_1), \varphi(e_2)] = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = \varphi(e_2) = \varphi([e_1, e_2])$$

$\Rightarrow$ φ is a lin. alg. homom which is a bijection (on its image)

$\Rightarrow A \simeq \{x \in gL(2, \mathbb{C}) \mid x_{21} = x_{22} = 0\}$

Proof: (in a special case)

If $\dim A < \infty$, $gL(A) \simeq gL(n, \mathbb{C})$

with basis $e_1 \ldots e_n$ (over $\mathbb{C}$)

$B = \mathrm{Der}(A)$ is LA (by thm 8)

$\varphi = \mathrm{ad}$, $\mathrm{ad} : A \longrightarrow \mathrm{Der}(A)$ a LA homom (by thm 14)

$\mathrm{Ker}(\mathrm{ad}) = Z(A)$ (by thm 15)

The special case :

Suppose $Z(A) = \{0\}$.

Then $A = A/\{0\} = A/Z(A) = A/\mathrm{Ker}(\mathrm{ad}) \cong \mathrm{Im}(\mathrm{ad}) \overset{\text{a Lie subalgebra}}{\subset} \mathrm{Der}(A)$

$\overset{\text{as a subA}}{\subset} gL(A) \simeq gL(n, \mathbb{C})$ $\Rightarrow$ A is isomorphic to a subalgebra of $gL(n, \mathbb{C})$

■

Algèbre de Lie :

le 30. 10. 19

Representations :

Rem : If V is a vector space, then (by thm 2) $gl(V) = (\text{End } V)^-$ (the Lie bracket $[\varphi, \psi] = \varphi \circ \psi - \psi \circ \varphi$) is a LA.

Déf : Let A be a LA (over $\mathbb{F}$) and V be a vector space (over $\mathbb{F}$).

A *representation of A* on V is a Lie alg. homom. $\rho : A \to gl(V)$

(That is, for $x \in A$, then $\rho(x)$ is a linear operator and the map $x \longmapsto \rho(x)$ is linear and $\rho([x,y]) = [\rho(x), \rho(y)]$)

Déf : a) *$\dim \rho$* $= \dim V$, if $\dim \rho < \infty$ then ρ is a *finite-dimensional representation*

b) ρ is a *faithful representation* if $\text{Ker}\,\rho = \{0\}$

Ex : The trivial rep. $\rho : \begin{array}{l} A \longrightarrow \mathbb{F} \\ x \longmapsto 0 \end{array}$

Ex : $A = gl(n, \mathbb{C})$, $\rho(x) = tr(x)$, $\rho : A \to \mathbb{C}$ is a one-dimention. repr.

Ex : A is a subalg. of $gl(n, \mathbb{C})$, $V = \mathbb{C}^n$

the natural rep. of A : $\rho(x) = x \in \text{Mat}(n, \mathbb{C})$

faithfull.

Ex : A : $[e_1, e_2] = e_2$ over $\mathbb{C}$, $V = \mathbb{C}^2$

$\rho(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\rho(e_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \rho : A \longrightarrow gl(\mathbb{C}^2)$

is a linear map. two-dim. repr.

Ex : A : $[e_1, e_2] = e_2$ avec $\mathbb{C}$, $V = \mathbb{C}[z]$ polynomials in z.

let $\rho : A \to gl(V)$ be a linear map such that :

$(\rho(e_1)p)(z) = z\,p'(z)$

$(\rho(e_2)p)(z) = z\,p(z)$

a basis of V : $v_0 = 1$, $v_1 = z$, $v_2 = z^2$, ...

$$[\rho(e_1), \rho(e_2)]\, z^n = \rho(e_1)\rho(e_2)\, z^n - \rho(e_2)\rho(e_1)\, z^n$$

$$= \rho(e_1)(z^{n+1}) - \rho(e_2)(n z^n)$$

$$= (n+1) z^{n+1} - n z^{n+1} = z^{n+1} = \rho(e_2)\, z^n$$

$$= \rho([e_1, e_2])\, z^n$$

$\Rightarrow$ ρ is a LA homom $\Rightarrow$ ρ is an infinite-dim repr of **A**.

Def: The adjoint representation of a LA A is the linear map

$\rho : A \longrightarrow gl(V) = A$, given by $\rho(x) = ad_x = [x, \cdot]$

$\dim \rho = \dim A$

Ex: $A = sL(2, \mathbb{C})$, $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$

$\dim A = 3$

$A \simeq \mathbb{C}^3$ as a vector space $e = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $h = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\left.\begin{aligned} ad_e(e) &= 0 \\ ad_e(h) &= -2e \\ ad_e(f) &= h \end{aligned}\right\} \Rightarrow ad_e = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left.\begin{aligned} ad_h(e) &= 2e \\ ad_h(h) &= 0 \\ ad_h(f) &= -2f \end{aligned}\right\} \Rightarrow ad_h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$ad_f = \left(\quad ? \quad \right)$$ exercise !

Thm 18:

a) the adjoint repr of a LA A is faithful $\Leftrightarrow$ $Z(A) = \{0\}$

b) every repr. of a simple LA is either trivial or faithful

le 30.10.19

Algèbre de Lie:

Proof: a) by th 15, $\mathrm{Ker}(\mathrm{ad}) = Z(A)$, $\rho = \mathrm{ad}$ is faithful

$$\Leftrightarrow \quad \mathrm{Ker}(\mathrm{ad}) = \{0\} \quad \Leftrightarrow \quad Z(A) = \{0\}$$

b) by th 15, $\mathrm{Ker}\,\rho$ is an ideal of A

$\Rightarrow$ either $\mathrm{Ker}\,\rho = \{0\} \Rightarrow \rho$ is faithful

or $\mathrm{Ker}\,\rho = A \Rightarrow \rho$ is the trivial repr.

▨

---

le 06.11.19

Modules:

Déf: Let A be a LA over $\mathbb{K}$. An A-module is a vector space V over $\mathbb{K}$ together with a bilinear map $A \times V \longrightarrow V$, $(x, v) \longmapsto x \cdot v$ such that $[x, y] \cdot v = x(y \cdot v) - y(x \cdot v)$ $\forall\, x, y \in A$, $v \in V$

Thm 19: "an A-module is a representation of A."

a) Let A be a LA and V a A-module. Then the map $\rho : A \to gl(V)$ given by $\rho(x)v = xv$ is a representation of A on V.

b) Let A be a LA and ρ a representation of A on V a vector space. Then, the map $A \times V \to V$ given by $x \cdot v = \rho(x)v$ defines the structure of an A-module on V.

Proof:

a) $(x, v) \longmapsto x \cdot v$ is bilinear, therefore:

$$\rho(x)(\lambda v_1 + \mu v_2) = x(\lambda v_1 + \mu v_2) = \dots = \lambda \rho(x) v_1 + \mu \rho(x) v_2$$

$\Rightarrow \rho(x) \in gl(V)$

$$\rho(\lambda x_1 + \mu x_2) v = (\lambda x_1 + \mu x_2) v = \dots = (\lambda \rho(x_1) + \mu \rho(x_2)) v$$

$\Rightarrow$ the map $x \longmapsto \rho(x)$ is linear $\forall v \in V$

$\rho([x,y])v = [x,y]v = x(yv) - y(xv)$

$= \ldots = [\rho(x), \rho(y)]\,x$

$\Rightarrow \rho([x,y]) = [\rho(x), \rho(y)]$

$\Rightarrow \rho$ is a representation of A on V.

b) Very similar, easier !

Ex : Let $A = \mathfrak{h}_3^+(\mathbb{C}) = \{(\nabla)\}$

Recall : $[e_{12}, e_{23}] = e_{13}$, $[e_{12}, e_{13}] = [e_{23}, e_{13}] = 0$

Let V be an infinite - dimentional vect. space with basis $\mathcal{B} = \{v_0, v_1, \ldots\}$

Consider a bilinear map $A \times V \longrightarrow V$ s.t, for $n \in \mathbb{N}$,

$$\begin{cases} e_{12} v_n = n\, v_{n-1} & (e_{12} v_0 = 0) \\ e_{23} v_n = v_{n+1} \\ e_{13} v_n = v_n \end{cases}$$

Let's check the bracket property for basis vectors :

$$e_{12}(e_{13} v_n) - e_{13}(e_{12} v_n) = e_{12} v_n - e_{13}(n v_{n-1}) = n v_{n-1} - n v_{n-1} = 0$$
$$= [e_{12}, e_{13}]\, v_n.$$

same for $[e_{23}, e_{13}]$

$$e_{12}(e_{23} v_n) - e_{23}(e_{12} v_n) = e_{12} v_{n+1} - e_{23}\, n v_{n-1}$$
$$= (n+1) v_n - n v_n = v_n = e_{13} v_n = [e_{12}, e_{23}]\, v_n$$

$\Rightarrow V$ is an A-module $\xleftrightarrow{\text{correspond}}$ $\rho : A \longrightarrow gl(V)$ a representation.

Let $V = \mathbb{C}[z]$ with canonical basis $\mathcal{B} = \{1, z, z^2, \ldots\}$

Take : $\rho(e_{12}) = \frac{\partial}{\partial z} \rightsquigarrow \frac{\partial}{\partial z} z^n = n\, z^{n-1}$

$\rho(e_{23}) = z \rightsquigarrow z \cdot z^n = z^{n+1}$

$\rho(e_{13}) = 1 \rightsquigarrow 1 \cdot z^n = z^n$

we have $[\frac{\partial}{\partial z}, z] = 1$ okay !

le 06.11.19

Algèbre de Lie :

Def: Let A be a LA and V an A-module. A submodule W of V is an invarient vector subspace,

ie : $x \cdot v \in W \quad \forall x \in A, v \in W$

$\Leftrightarrow \quad \rho(x)v \in W \quad \forall x \in A, v \in W$

Rem: $\{0\}$ and V are always submodule of V.

Ex: Let $A = \mathfrak{h}^+_3(\mathbb{C})$.

Recall : $[e_{12}, e_{23}] = e_{13}$, $[e_{13}, e_{12}] = [e_{13}, e_{23}] = 0$

Let ρ be the natural representation of A on $V = \mathbb{C}^3$

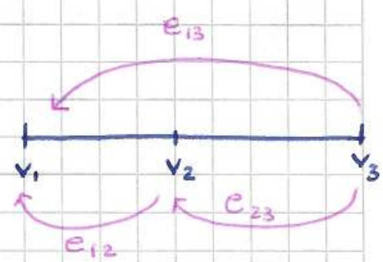
$$e_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Take $\mathcal{B} = \left\{ v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ the canonical basis of $V = \mathbb{C}^3$

$e_{12} v_1 = 0 \qquad e_{12} v_2 = v_1 \qquad e_{12} v_3 = 0$

$e_{23} v_1 = 0 \qquad e_{23} v_2 = 0 \qquad e_{23} v_3 = v_2$

$e_{13} v_1 = 0 \qquad e_{13} v_2 = 0 \qquad e_{13} v_3 = v_1$



$W_1 = \text{span}(v_1)$ and $W_2 = \text{span}(v_1, v_2)$ are submodulus of V

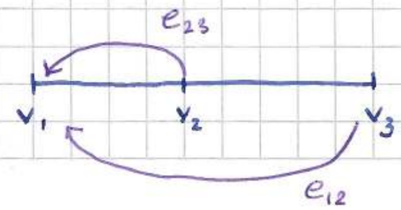
Now consider the adjoint representation ρ of A on $V = A$ (instead of the natural one)

The basis of A is $\mathcal{B} = \{ v_1 = e_{13}, v_2 = e_{12}, v_3 = e_{23} \}$

$e_{13} v_1 = 0 \qquad e_{13} v_2 = 0 \qquad e_{13} v_3 = 0$

$e_{12} v_1 = 0 \qquad e_{12} v_2 = 0 \qquad e_{12} v_3 = v_1$

$e_{23} v_1 = 0 \qquad e_{23} v_2 = -v_1 \qquad e_{23} v_3 = 0$



$W_1 = \text{span}(v_1)$, $W_2 = \text{span}(v_1, v_3)$ and $W_3 = \text{span}(v_1, v_2)$ are submodulus of V.

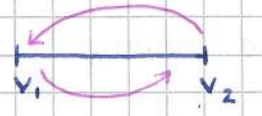
Rem : Caution : A submodule can be the linear span of linear combinations of basis vectors.

Ex : Let $A = \text{span}(e_1, e_2)$ s.t $[e_1, e_2] = 0$.

Consider the representation ρ on $V = \mathbb{C}^2$ s.t. $\rho(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\rho(e_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and take $\mathcal{B}_V = \left\{ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$e_1 v_1 = v_1$, $e_1 v_2 = v_2$

$e_2 v_1 = v_2$, $e_2 v_2 = v_1$



$W_1 = \text{span}(e_1 + e_2)$ and $W_2 = \text{span}(e_1 - e_2)$ are submodules of V.

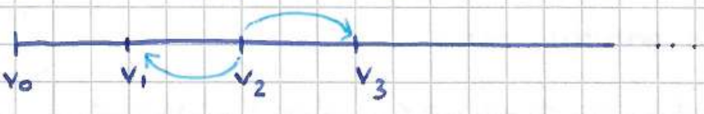
Rem : Let ρ be the adjoint representation of a LA A.

A vector subspace $I \subset A$ is invariant $\Leftrightarrow$ I is an ideal of A. $x \cdot y = \rho(x) y = \text{ad}_x(y) = [x, y] \in I$ for $x \in I$, $y \in A$

Déf : Let A be a LA. An A-module V is simple if $V \neq \{0\}$ and its only submodules are $\{0\}$ and V.

The corresponding representation ρ is irreducible.

Ex :
1. Any one-dimensional A-module is simple.
2. Let $A = h_3^+(\mathbb{C})$ and $\rho : A \to gl(V)$, $V = \mathbb{C}[z]$, s.t $\rho(e_{12}) = \frac{\partial}{\partial z}$, $\rho(e_{23}) = z$, $\rho(e_{13}) = 1$



is a simple module

Def : Let A be a LA and V be an A-module, let $v \in V$

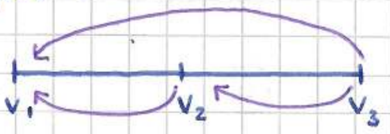
a) We denote $W_v := \text{span}(x_1 x_2 \quad x_m v \mid m \geq 0, x_i \in A)$

b) v is said to be cyclic if $W_v = V$.

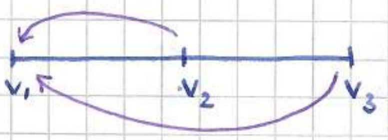
le 06.11.19

Algèbre de Lie :

Ex : Let $A = h_3^+(\mathbb{C})$. We take the same two representations as before :



v_3 is cyclic !



no cyclic vector.

Thm 20 :

a) For every $v \in V$, the subspace W_v is a submodule of V.

b) V is simple $\Leftrightarrow$ every $v \in V$ is cyclic.

Proof :

a) Let $u \in W_v$. Then, $u = \alpha v + \sum_i \alpha_i x_i v + \sum_{i,j} \alpha_{ij} x_i (x_j v) + \ldots$

$x \in A$

$$\Rightarrow x u = \alpha \underbrace{x v}_{\in W_v} + \sum_i \alpha_i \underbrace{x(x_i v)}_{\in W_v} + \sum_{i,j} \alpha_{ij} \underbrace{x(x_i(x_j v))}_{\in W_v} + \ldots \in W_v$$

b) $\Rightarrow$: Take $v \in V$, then W_v is a submodule ($W_v \neq \{0\}$)

$$\overset{\text{simple}}{\Rightarrow} W_v = V \Rightarrow v \text{ is cyclic.}$$

$\Leftarrow$: Suppose V is not simple

$\Rightarrow$ There is $W \subsetneq V$ a proper submodule

Take $v \in W$, $u \notin W$. v is cyclic $\Rightarrow u \in W_v$

$$\Rightarrow u = \alpha \underbrace{v}_{\in W} + \sum_i \alpha_i \underbrace{x_i v_i}_{\in W} + \ldots \quad \in W \quad ↯$$

$\Rightarrow V$ is simple

▨

lc 13.11.19

Rem :

	A is simple	A is not simple
V is simple	①	②
V is not simple	③	④

① ρ is the adjoint rep. of a simple LA

(a submodule $\Leftrightarrow$ an ideal)

② $A = gl(2, \mathbb{C})$, ρ is the natural rep. of $V = \mathbb{C}^2$

③ $A = sl(2, \mathbb{C})$, $V = \mathbb{C}^3$, $\rho(e) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\rho(f) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\rho(h) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$V_1 = \mathrm{Span}(v_1, v_2)$; $V_2 = \mathrm{Span}(v_3)$ [submodule]
↳ not simple

④ A is an abelian LA : $[e_1, e_2] = 0$, $\rho(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\rho(e_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$V_1 = \mathrm{Span}(v_1)$; $V_2 = \mathrm{Span}(V_2)$ not simple

$\Rightarrow$ no relation between a LA and module being simple

Rem : In ③ and ④ , $V = V_1 \oplus V_2$

Déf : Let V be a vect. space and V_1, V_2 be vector subspaces. Then V is the direct sum of V_1 and V_2 ($V = V_1 \oplus V_2$) if every $v \in V$ is uniquely decomposed as $v = v_1 + v_2$, $v_1 \in V_1$, $v_2 \in V_2$

Déf : Let A be a LA and V be an A-module , V is semisimple if $V = V_1 \oplus V_2 \oplus \ldots = \bigoplus_{i=1}^{n} V_i$ where V_i is a simple submodule le 13. 11. 19

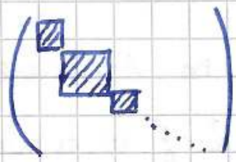
Algèbre de lie :

Def: If V is semisimple, then the corresponding representation is completely reducible.

Rem: If ρ is a finite-dimentional completely reducible rep. ($\rho : A \to V$, $\dim V < \infty$) then there is a basis of V such that $\rho(x)$ is a block-diagonal matrix $\forall x \in A$.

Take a basis $\mathcal{E} = \{ \underbrace{e_1, \dots, e_m}_{\text{basis of } V_1}, \underbrace{e_{m+1}, \dots, e_k}_{\text{basis of } V_2} \dots \}$

$$V = \bigoplus_{i=1}^{n} V_i$$

then $\rho(x) =$  $x \in A$

Def: A $\mathcal{L}A$ A is said to be reductive if it's adjoint repr is completely reducible.

Ex: a) Every simple LA

b) Every abelian algebra

$\mathcal{E} = \{ e_1, \dots, e_k \}$, $V_i = \text{Span}(e_i)$, $V = A = \bigoplus_{i=1}^{k} V_i$

c) $A = gl(n, \mathbb{C}) = SL(2, \mathbb{C}) \oplus \text{Span}(E)$

$x \in gl(n, \mathbb{C})$, $x = \underbrace{x - \frac{1}{n} tr(x) \cdot E}_{\in SL(n, \mathbb{C})} + \frac{1}{n} tr(x) \cdot E$

Def: Let A be a LA. An A-module V is indicomposable if it has no non-zero submodules V_1 and V_2 s.t $V = V_1 \oplus V_2$.

Ex: Every simple A-module is indicomposable.

Ex: A, $[e_1,e_2] = e_2$, $\rho(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\rho(e_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$V_1 = \text{Span}(v_1)$ is a submodule but $V \neq V_1 \oplus \ldots$

$\text{Hom}_A(V_1,V_2)$:

Def: Let A be a LA (over $\mathbb{F}$), let V_1 and V_2 be A-modules. A linear map $f: V_1 \to V_2$ is a homomorphism of A-modules if $f(xv) = x f(v)$ $\forall x \in A$ $\forall v \in V_1$

Def: $\text{Hom}_A(V_1,V_2)$ is the set / a vector space of all homomorphism of A-modules V_1 and V_2

Thm 21: Let A be a LA and V_1, V_2 be A-modules. Let $f \in \text{Hom}_A(V_1,V_2)$ then $\ker f$ is a submodule in V_1 and $\text{Im} f$ is a submodule in V_2.

Pf: Let $v \in \text{Ker} f$, then $f(xv) = xf(v) = 0 \Rightarrow x \cdot v \in \text{Ker} f$

$\Rightarrow \text{Ker} f$ is an invariant vect. subs. (= a submodule)

Let $u \in \text{Im} f$, then $\exists v \in V_1$ s.t $f(v) = u \Rightarrow x \cdot u = xf(v) = f(xv)$

$x \cdot u \in \text{Im} f \Rightarrow \text{Im} f$ is an invariant subspace.

$\boxtimes$

Def: Let A be a LA. Let $\rho_1 : A \to gl(V_1)$, $\rho_2 : A \to gl(V_2)$ be repres., then ρ_1 and ρ_2 are isomorphic repr. if there exists a bijective homomorphism of the A-modules V_1 and V_2.

Rem: Let ρ_1 and ρ_2 be finite-dimentional isomorphic representation then $V_1 \cong V_2 \cong V$ (as a vector space). Let $f: V_1 \to V_2$ be a bijective hom then $f \in gl(V)$, so f is represented (if we choose a basis of V) by an invertible matrix F.

$f(x \cdot v) = x f(v) \Rightarrow \quad F \cdot \rho_1(x) v = \rho_2(x) F v \quad \forall x \in A \quad \forall v \in V_1$

equivalent definition.

$$\Leftrightarrow \quad F\rho_1(x) = \rho_2(x) F \quad \forall x \in A \quad \Leftrightarrow \quad F\rho_1(x)F^{-1} = \rho_2(x) \quad \forall x \in A$$

Algèbre de Lie :

Ex : A : $[e_1, e_2] = e_2$, $V = \mathbb{C}^2$

$\rho_1(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\rho_1(e_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\rho_2(e_1) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\rho_2(e_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Suppose ρ_1 and ρ_2 are isomorphic :

$F\rho_1(e_1) \cdot F^{-1} = \rho_2(e_1)$

$Tr(F\rho_1(e_1)F^{-1}) = Tr(\rho_1(e_1)) \neq Tr(\rho_2(e_1))$ ↯

$Dét(F\rho_1(e_1)F^{-1}) = Dét(\rho_1(e_1)) \neq Dét(\rho_2(e_1))$ ↯

$\Rightarrow$ ρ_1 and ρ_2 are not isomorphic.

Thm 22 : Let A be a LA (over $\mathbb{F}$). Let V_1 and V_2 be simple A-modules. Let $f \in Hom_A(V_1, V_2)$ then $f = 0$ or f is an isomorphism.

Pf : By thm 21, $Ker f$ is a submodule in V_1

$\Rightarrow$ either $Ker f = V_1 \Rightarrow f = 0$

or $Ker f = \{0\} \Rightarrow Im f \neq \{0\}$ and by thm 21

$Im f$ is a submodule in $V_2 \Rightarrow Im f = V_2$

$\Rightarrow f$ is bijective $\Rightarrow f$ is an isomorphism.

▨

Thm 23 : (Schur's Lemma)

Let A be a LA over $\mathbb{C}$. Let ρ be a finite-dimensional irreducible rep. of A on a vect. space V. Let $f \in gl(V)$ such that $f \cdot \rho(x) = \rho(x) \cdot f$ $\forall x \in A$.

Then there exists $\lambda \in \mathbb{C}$ s.t $f \equiv \lambda \cdot id$.

(Equivalently, if A is a finite-dim simple A-module and $f \in Hom_A(V)$, $\exists \lambda \in \mathbb{C}$ s.t. $f \equiv \lambda \cdot id$)

Pf: φ is represented by a complex matrix F.

Then $\exists\, u \in V$ and $\exists\, \lambda \in \mathbb{C}$ s.t $F \cdot u = \lambda \cdot u$

Set $W = \{ v \in V \mid Fv = \lambda v \}$, $W \neq \{0\}$

Take $v \in W$, $x \in A$, $F\rho(x)v = \rho(x)\,Fv = \rho(x)\lambda v = \lambda\,\rho(x)v$

$\Rightarrow \rho(x) \cdot v \in W \Rightarrow W$ is a submodule of V $\Rightarrow W = V$

$\Rightarrow F = \lambda E \Leftrightarrow \varphi = \lambda \cdot id$

▨

Thm 24:

Let ρ be a finite-dim. irreducible rep. of a complex LA A.

Let $w \in Z(A)$, then $\rho(w) = \lambda \cdot id$

Pf: Set $\varphi = \rho(w)$ take $x \in A$, $\varphi \cdot \rho(x) - \rho(x)\varphi = [\rho(w), \rho(x)]$

$$= \rho(\underbrace{[w, x]}_{0}) = 0.$$

$\Rightarrow \forall x \in A \quad \varphi \cdot \rho(x) = \rho(x) \cdot \varphi \Rightarrow \varphi = \lambda \cdot id$

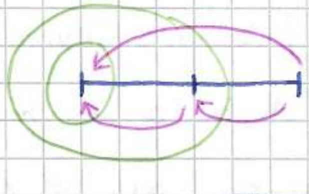
▨

Ex: $A = n^+(3, \mathbb{C})$ $[e_{12}, e_{23}] = e_{13}$, $[e_{13}, e_{23}] = [e_{12}, e_{13}] = 0$

Let ρ be the natural repr.

Then $\rho(e_{13}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ but $e_{13} \in Z(A)$

$\Rightarrow \rho$ is not irreducible



Le 20. 11. 19

The Killing form:

Def: Let A be a LA. A bilinear form $B: A \times A \to \mathbb{F}$.

a) B is symmetric if $B(x,y) = B(y,x) \quad \forall x, y \in A$

b) B is invariant if $B([x,y],z) = B(x,[y,z]) \quad \forall x, y, z \in A$

Algèbre de Lie : Le 20.11.19

Def : Let B be a symmetric bilinear form on a LA A.

a) Let $W \subset A$ then $W^{\perp} = \{ x \in A \mid B(x,y) = 0 \ \forall y \in W \}$

Rem : In general $W \cap W^{\perp} \neq \emptyset$.

b) $\operatorname{Ker} B = A^{\perp} = \{ x \in A \mid B(x,y) = 0 \quad \forall y \in A \}$

c) B is non-degenerate if $\operatorname{Ker} B = \{0\}$

Thm 25 : Let B be an invariant symm. bilinear form on a LA A.

a) Let $W \subset A$. Then $W^{\perp}$ is a vector subspace of A.

b) Let $I \subset A$ be an ideal. Then $I^{\perp}$ is an ideal of A.

Proof :

a) Let $x \in W$ and $y, z \in W^{\perp} \Rightarrow B(x, \lambda y + \mu z) = \lambda B(x,y) + \mu B(x,z) = 0 \Rightarrow W^{\perp}$ is a subspace.

b) Let $x \in I^{\perp}$, take $y \in A$ and $z \in I$

$$B([x,y],z) = B(x, \underbrace{[y,z]}_{\in I}) = 0 \qquad \forall z \in I$$

$$\Rightarrow [x,y] \in I^{\perp} \qquad \forall y \in A$$

▨

Rem : Let A be a simple LA. Let B be an invariant symm. bilinear form on A. Then either $B = 0$ or B is non-degenerate

(Indeed : $\operatorname{Ker} B$ is an ideal of A

$\Rightarrow \operatorname{Ker} B = A \Rightarrow B = 0$

or $\Rightarrow \operatorname{Ker} B = \{0\} \Rightarrow B$ is non-deg)

Thm 26 : Let A be a LA. Let $\rho : A \longrightarrow gl(V)$ be a finite-dimensional rep. of A. Set $B(x,y) = Tr(\rho(x)\rho(y))$, then B is an invariant symm. bilinear form of A.

Proof: B is symmetric: $Tr(ab) = Tr(ba)$

B is bilinear: $x \mapsto \rho(x)$ is a linear map ant Tr is linear.

Invariant: $B([x,y],z) = Tr(\rho([x,y])\rho(z))$

$$= Tr(([\rho(x),\rho(y)])\rho(z))$$
$$= Tr((\rho(x)\rho(y) - \rho(y)\rho(x))\rho(z))$$
$$= Tr(\rho(x)\rho(y)\rho(z) - \rho(x)\rho(y)\rho(z))$$
$$= Tr(\rho(x)(\rho(y)\rho(z) - \rho(z)\rho(y)))$$
$$= Tr(\rho(x)(\rho([y,z])))$$
$$= B(x,[y,z])$$

▨

Ex: $A = gl(n,\mathbb{C})$. Let ρ be the natural repr.

$B(x,y) = Tr(x,y)$

B is not degenerate: $x \in A$, take $y = x^* = \bar{x}^t$

$B(x,y) = Tr(xx^*) = \sum_{i,j=1}^{n} x_{ij}(x^*)_{ji} = \sum_{i,j=1}^{n} |x_{ij}|^2 = 0$

$\Leftrightarrow x = 0$

Def: Let A be a LA. Let $\rho = ad.$ be the adjoint repr. of A.

Then $K(x,y) = Tr(ad_x\, ad_y)$ is the Killing form of A.

Rem: If $Z(A) \neq \{0\}$, then $Z(A) \subset Ker\, K \neq \emptyset \Rightarrow K$ is degenerate

Ex: $A = gl(n,\mathbb{C}) \Rightarrow Z(A) \neq \{0\}$ so K is degenerate

Thm 27: Let A be a complex nilpotent LA.

Then $K = 0$. $(K(x,y) = 0 \ \forall x,y \in A)$

Proof: Suppose $K \neq 0 \Rightarrow \exists x \in A$ s.t $K(x,x) \neq 0$.

(Indeed: if $K(x,x) = 0 \ \forall x \in A \Rightarrow K(x+y, x+y)$

$= K(x,x) + 2K(x,y) + K(y,y) \Rightarrow K(x,y) = 0 \ \forall x,y \in A$

$\Rightarrow K = 0$ })

Algèbre de Lie: le 20.11.19

take $x \in A$ s.t $K(x,x) \neq 0 \Rightarrow tr(ad_x ad_x) \neq 0$

$\Rightarrow ad_x ad_x$ has at least one eigenvector y with eigenvalue $\lambda \neq 0$

$\Rightarrow ad_x ad_x (y) = \lambda y$, $\lambda \neq 0$ $(\Leftrightarrow$ $\underset{\in A_{(2)}}{[x,[x,y]]} = \lambda y \Rightarrow \underset{\in A_{(1)}}{[x,y]} \neq 0$

$\Rightarrow [x,[x,\overset{\in A_{(3)}}{[x,y]}]] = \lambda [x,y] \Rightarrow \underset{\in A_{(4)}}{[x[x,[x,[x,y]]]]} = \lambda^2 y$

$\Rightarrow A_{(k)} \neq \{0\} \quad \forall k$

$\Rightarrow A$ is not nilpotent ↯

▨

Thm 28: (Cartan's first criterion)

A LA A is solvable $\Leftrightarrow$ $A' \subset Ker\, K$

(ie: $K(x,y) = 0 \quad \forall x \in A' \quad \forall y \in A$)

Ex: A: $[e_1,e_2] = e_2$; $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\end{pmatrix}$, $A' = Span(e_2)$

$ad_{e_1} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \qquad ad_{e_2} = \begin{pmatrix}0 & 0\\-1 & 0\end{pmatrix}$

$x \in A$, $y \in A'$, $ad_x = \begin{pmatrix}0 & 0\\ * & *\end{pmatrix}$, $ad_y = \begin{pmatrix}0 & 0\\ * & 0\end{pmatrix}$

$Tr(ad_x \cdot ad_y) = Tr\begin{pmatrix}0 & 0\\ * & 0\end{pmatrix} = 0$

$\Rightarrow A' \subset Ker\, K \quad \Rightarrow \quad A$ is solvable

$$\mathcal{K} = \begin{pmatrix}K(e_1,e_1) & K(e_1,e_2)\\ K(e_2,e_1) & K(e_2,e_2)\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

↑ The Killing matrix.

The Casimir Operator:

Lemma: Let $B: V \times V \longrightarrow \mathbb{C}$ be a symm. bilinear form on a vect. space V. Let $\mathcal{B}$ be it's matrix ($\mathcal{B}_{ij} = B(e_i,e_j)$)

Let $e_1, \dots, e_n$ be a basis of V.

B is non-degenerate $\Leftrightarrow$ $\det \mathcal{B} \neq 0$

Proof: $\mathcal{B} = \begin{pmatrix} B(e_1,e_1) & \cdots & B(e_1,e_n) \\ \vdots & \ddots & \vdots \\ B(e_n,e_1) & \cdots & B(e_n,e_n) \end{pmatrix}$

If $\det \mathcal{B} = 0 \Rightarrow$ The rows of $\mathcal{B}$ are lin. depend.

$\exists \lambda_i \in \mathbb{C}$ st $0 = \sum_{i=1}^{n} \lambda_i B(e_i, e_k) \ \forall k \Leftrightarrow B(\underbrace{\sum_{i=1}^{n} \lambda_i e_i}_{= v \in V} e_k) = 0 \ \forall k$

$\Rightarrow v \in \operatorname{Ker} B$

▨

Def: Let A be a LA. Let $\mathcal{E} = \{e_1, \dots, e_k\}$ be a basis of A. Suppose that the Killing of A is non-degenerate ($\det \mathcal{K} \neq 0$)

Set $e_i^* = \sum_{j=1}^{n} (\mathcal{K}^{-1})_{ij} e_j$, $\mathcal{E}^* = \{e_1^*, \dots, e_k^*\}$ is the dual basis of A.

Rem: $K(e_i^*, e_k) = K(\sum_{j=1}^{n} (\mathcal{K}^{-1})_{ij} e_j, e_k) = \sum_{j=1}^{n} (\mathcal{K}^{-1})_{ij} K(e_j, e_k)$
$= ((\mathcal{K}^{-1})\mathcal{K})_{ik} \overset{\blacklozenge}{=} \delta_{ik}.$

Def: Let $\rho: A \to gl(V)$ be a finite dimensional repr of A, then $C \in gl(V)$ given by $C = \sum_{i=1}^{n} \rho(e_i)\rho(e_i^*)$ is the Casimir operator (associated to ρ).

Thm 29: a) $C \in \operatorname{Hom}_A(V,V)$ $(C \cdot \rho(x) = \rho(x) C \ \ \forall x \in A)$

b) if $\rho = ad$, then $tr C = n$

Proof:

a) $[a \cdot b, c] = abc - cab = abc - acb + acb - cab \overset{\bullet}{=} a[b,c] + [a,c]b$

Réécrire

$[e_i; e_j] = \sum_{m=1}^{n} c_{ij}^{m} e_m$ le 27.11.19

$[e_i^*, e_k] = \sum_{m=1}^{n} \tilde{c}_{ik}^{m} e_m^*$

$c_{ik}^{m} \overset{\blacklozenge}{=} K([e_i, e_k], e_m^*) = K(e_i, [e_k, e_m^*]) =$
$= -K(e_i, [e_m^*, e_k]) \overset{\blacklozenge}{=} -\tilde{c}_{mk}^{i}$

$C\rho(e_k) - \rho(e_k) C = [C, \rho(e_k)] \overset{\bullet}{=} \sum_{i=1}^{n} (\rho(e_i) - [\rho(e_i^*), \rho(e_k)])$
$+ [\rho(e_i), \rho(e_k)] \rho(e_i^*)) = \sum_{i=1}^{n} \rho(e_i)\rho([e_i^*, e_k]) + \rho([e_i, e_k])\rho(e_i^*)$

le 27.11.19

Algèbre de Lie:

Proof:

a) $[a \cdot b, c] = abc - cab - acb + acb$

$$= a[b,c] + [a,c]b \quad (*)$$

$$[e_i, e_j] = \sum_{m=1}^{n} c_{ij}^{m} e_m$$

$$[e_i^*, e_j] = \sum_{m=1}^{n} \tilde{c}_{ij}^{m} e_m^* \leftrightarrow [e_m^*, e_K] = \sum_{i=1}^{n} \tilde{c}_{mK}^{i} e_i^*$$

$$c_{iK}^{m} \overset{\blacklozenge}{=} K([e_i, e_K], e_m^*) = K(e_i, [e_K, e_m^*])$$

$$= -K(e_i, [e_m^*, e_K]) \overset{\blacklozenge}{=} -\tilde{c}_{mK}^{i} \quad (*)$$

$$c\rho(e_K) - \rho(e_K)c = [c, \rho(e_K)]$$

$$\overset{(*)}{=} \sum_{i=1}^{n} (\rho(e_i)[\rho(e_i^*), \rho(e_K)] + [\rho(e_i), \rho(e_K)]\rho(e_i^*))$$

$$= \sum_{i=1}^{n} (\rho(e_i)\rho([e_i^*, e_K]) + \rho([e_i, e_K])\rho(e_i^*))$$

$$= \sum_{i,m=1}^{n} (\underbrace{\tilde{c}_{iK}^{m} \rho(e_i)\rho(e_m^*)}_{= \boxed{\tilde{c}_{mK}^{i}} \rho(e_m)\rho(e_i^*)} + \boxed{c_{iK}^{m}} \rho(e_m)\rho(e_i^*)) \overset{(*)}{=} 0$$

$$\Rightarrow c\rho(x) = \rho(x) \cdot c \quad \forall x \in A$$

b) $\rho = ad$

$$\mathrm{tr}\, C = \sum_{i=1}^{n} (ad_{e_i} - ad_{e_i^*}) = \sum_{i=1}^{n} \underbrace{K(e_i, e_i^*)}_{=1} = n = \dim A$$

▨

Recall: $K(x,y) = \mathrm{tr}(ad_x\, ad_y)$; $\mathring{K}_{ij} = K(e_i, e_j)$

$$e_i^* = \sum_{j=1}^{n} (\mathring{K}^{-1})_{ij} e_j$$

$$K(e_i, e_\ell^*) = \delta_{i\ell}$$

$$C = \sum_{i=1}^{n} \rho(e_i)\rho(e_i^*) \in gl(V)$$

Representations of $sl(2,\mathbb{C})$:

A: $[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$

(-1) The natural representation (= the fundamental representation)

on $V = \mathbb{C}^2$, a basis : $v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\begin{tikzcd} v_0 \arrow[r, bend right, "f"'] & v_1 \arrow[l, bend right, "e"'] \end{tikzcd} \Bigg\} *$$

$\rho(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\rho(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\rho(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

* is a simple $sL(2,\mathbb{C})$ - module

(0) The trivial representation

on $V = \mathbb{C}$, a basis : $v_0 = 1$, $\rho(x) = 0$

v_0 is a simple $sL(2,\mathbb{C})$ - module .

(ad)
$$\begin{tikzcd} e \arrow[r, bend right, "f"'] & h \arrow[l, bend right, "e"'] \arrow[r, bend right, "f"'] & f \arrow[l, bend right, "e"'] \end{tikzcd}$$
is a simple $sL(2,\mathbb{C})$ - module

(no proper invariant subspace)

$e \approx \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $h \approx \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $f \approx \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$ad_e = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $ad_h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $ad_f = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$

$e_1 = e$
$e_2 = h$
$e_3 = f$

$k = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$ $\quad (k_{ij} = K(e_i, e_j))$

$e_i^* = \sum_j (k^{-1})_{ij} e_j$ $\qquad k^{-1} = \begin{pmatrix} 0 & 0 & 1/4 \\ 0 & 1/8 & 0 \\ 1/4 & 0 & 0 \end{pmatrix}$

$\Rightarrow e^* = 1/4 f$; $h^* = \frac{1}{8} h$; $f^* = \frac{1}{4} e$

finite dim arbitrary rep.

(ρ) $E = \rho(e)$, $F = \rho(f)$, $H = \rho(h) \in gL(V)$ if $\rho : sL(2,\mathbb{C}) \to gL(V)$

rewriting Lie brackets.
$$\begin{cases} EF - FE = H \\ HE - EH = 2E \\ HF - FH = -2F \end{cases}$$

just to be more beautiful

$C = \textcircled{2} \sum_{i=1}^{3} \rho(e_i)\rho(e_i^*) =$

$= \frac{1}{2} EF + \frac{1}{2} FE + \frac{1}{4} H^2 =$

$= \frac{1}{2}(EF + H) + \frac{1}{2} FE + \frac{1}{4} H^2 = FE + \frac{1}{4} H(H+2)$

$= EF + \frac{1}{4} H(H-2)$

le 27.11.19

Algèbre de Lie :

Déf: Let V be an $sl(2,\mathbb{C})$ - module

a) $v \in V$, $v \neq 0$ is a weight vector if $Hv = \alpha \cdot v$, $\alpha \in \mathbb{C}$

b) $V_\alpha = \{ v \in V \mid H \cdot v = \alpha v \}$ is a weight subspace (of weight α)

c) $v \in V$, $v \neq 0$ is a heighest weight vector (hwv) of weight α if $Hv = \alpha v$ and $Ev = 0$

d) $v \in V$, $v \neq 0$ is a lowest w.v (L.w.v) of weight α if $Hv = \alpha v$ and $Fv = 0$.

Lemma : Let $v \in V_\alpha$, if $E \cdot v \neq 0$ then $Ev \in V_{\alpha+2}$

if $Fv \neq 0$ then $Fv \in V_{\alpha-2}$

Ex : In ⓪, ①, (ad)

$v_0 = 1$, $v_0 = \binom{1}{0}$, $v = e$ are h.w.v of weight $0, 1, 2$

Proof: $H(E \cdot v) = (HE)v = (EH + 2E)v = \alpha Ev + 2Ev = (\alpha+2) Ev$

$\Rightarrow E \cdot v \in V_{\alpha+2}$

(Repeat for F)

▨

Thm 30 : Let V be a simple $sl(2,\mathbb{C})$ - module, $\dim V < \infty$

a) V containes a unique h.w.v v_0 (up to a scalar multipe)

b) $Hv_0 = \lambda \cdot v_0$, where $\lambda + 1 = \dim V$.

c) vectors $v_k = F^k \cdot v_0$, $k = 0 \dots \lambda$ form a basis of V.

d) as a vector space, $V = \bigoplus_{k=0}^{\lambda} V_{\lambda-2k}$, $\dim V_{\lambda-2k} = 1 \ \forall k$

Proof:

$\dim V < \infty \Rightarrow H$ has at least one eigenvector.

ie: $\exists w_0 \in V$, $Hw = \alpha w_0$.

Set $w_k = E^k w_0 \quad k = 0, 1, \dots$

by the lemma $w_k \in V_{\alpha + 2k} \Rightarrow$ they are linearly independents

$\dim V < \infty \Rightarrow \exists m \in \mathbb{Z}_{\geq 0}$ s.t $w_m \neq 0$, $E w_m = 0$

Set $v_0 = w_m$,

$Hv_0 = \lambda v_0$, $Ev_0 = 0 \Rightarrow v_0$ is a h.w.v.

Set $v_k = F^k v_0$, by Lemma $\Rightarrow v_k \in V_{\lambda - 2k}$

$\Rightarrow v_k$ are lin. indep , $\dim V < \infty \Rightarrow \exists m \in \mathbb{Z}_{\geq 0}$ st $v_m \neq 0$ and $F v_m = 0$

$C =$ const. id on V (V is simple, thm 25) (Casimir Operator)

$\Rightarrow C \cdot v_0 = (FE + \frac{1}{4} H(H+2)) v_0 = \frac{1}{4} \lambda(\lambda+2) v_0$

$C \cdot v_m = (EF + \frac{1}{4} H(H-2)) Vm = \frac{1}{4}(\lambda - 2m)(\lambda - 2m - 2) Vm$

$$\Rightarrow 0 = \frac{1}{4}\left(\lambda(\lambda+2) - (\lambda-2m)(\lambda-2m-2)\right)$$

$$= \frac{1}{4}\left(\lambda(2+2m+2+2m) - 4m^2 - 4m\right)$$

$$= \lambda(m+1) - m(m+1) = \underbrace{(m+1)}_{\neq 0}(\lambda - m)$$

$\Rightarrow \lambda = m. \quad \Rightarrow \lambda \in \mathbb{Z}_{\geq 0}$

---

le 04.12.19

So we have $v_0, v_1, \dots, v_\lambda$ vectors, linearly independents.

$W = \text{Span}(v_0, \dots, v_\lambda)$

Lemma: W is a submodule of V

$Hv_k = (\lambda - 2k) v_k$, $F v_k = v_{k+1}$, $\Rightarrow W$ is invariant w.r.t H and F.

$Ev_0 = 0$, $Ev_{k+1} = EFv_k = (C - \frac{1}{4}H(H-2)) v_k = \frac{1}{4}\lambda(\lambda+2) v_k$
$- \frac{1}{4}(\lambda - 2k)(\lambda - 2k - 2) v_k = (\lambda - k)(k+1) v_k$

$\Rightarrow E v_k = k(\lambda + 1 - k) V_{k-1}$ (*)

$\Rightarrow W$ is invariant $\rho(x)$ $\forall x \in sl(2, \mathbb{C}) \Rightarrow W$ is a submodule ■

le 04.12.19

Algèbre de Lie :

V is simple , $W \neq \{0\} \Rightarrow V = W \Rightarrow$

v_k are linearly independant $\Rightarrow v_k = F^k v_0$ is a basis of V.

$\Rightarrow \dim V = \lambda + 1$

$$V = \bigoplus_{k=0}^{\lambda} \mathrm{Span}(v_k) = \bigoplus_{k=0}^{\lambda} V_{\lambda-2k} \text{ , where } \dim V_{\lambda-2k} = 1.$$

Suppose u is a h.w.v. in $V \Rightarrow H \cdot u = \alpha\, u \Rightarrow u = v_k$ for some k.

And $Eu = 0 \iff Ev_k = 0 \iff \begin{cases} k = 0 & \text{for some } k \\ \lambda + 1 - k = 0 & \nexists\ k \leq \lambda \end{cases}$

$\Rightarrow u = v_0 \Rightarrow$ the h.w.v is unique.

☒

Thm 31 :

a) For any $\lambda = 0, 1, 2, \dots$ there exists an $sl(2,\mathbb{C})$ - module (simple) $V[\lambda]$ of dimension $\lambda + 1$.

b) $V[\lambda]$ is generated by a h. w. v v_0 of weight λ

c) $\forall v \in V[\lambda]$, $c \cdot v = \frac{1}{4}\lambda(\lambda+2) \cdot v$, $\omega_\lambda = \frac{1}{4}\lambda(\lambda+2)$

d) $V[\lambda] = V[\tilde{\lambda}] \iff \omega_\lambda = \omega_{\tilde{\lambda}}$

e) if V is a simple $SL(2,\mathbb{C})$ - module of dim $\lambda + 1$, then $V \cong V[\lambda]$.

Proof :

b), a) Let $v_0, \dots, v_\lambda$ be a basis of a vector space $V[\lambda]$ of dim. $\lambda + 1$. Define the action of $sl(2,\mathbb{C})$ on $V[\lambda]$ by:

$h v_k = (\lambda - 2k) v_k$, $f v_k = v_{k+1}$, $e v_k = k(\lambda + 1 - k) v_{k-1}$.

$e(f v_k) - f(e v_k) = e v_{k+1} - f(k(\lambda+1-k) v_{k-1})$

$= ((k+1)(\lambda - k) - k(\lambda + 1 - k)) v_k = (\lambda - 2k) v_k \,{}_{\wedge}\!= [e,f]\, v_k$

$h(f v_k) - f(h v_k) = -2 f v_k = [h,f]\, v_k$ $\quad = h v_k$

$h(e v_k) - e(h v_k) = 2 e v_k = [h,e]\, v_k$

$\Rightarrow$ V[λ] is p 47 an sL(2,ℂ - module.

V$[\lambda]$ is simple because $e \cdot v_k \neq 0 \quad \forall k \neq 0$

$f \cdot v_k \neq 0 \quad \forall k \neq \lambda$

c) $(FE + \frac{1}{4} H(H+2))v_0 = \frac{1}{4}\lambda(\lambda+2)v_0$

d) $w_\lambda - w_{\tilde\lambda} = \frac{1}{4}\lambda(\lambda+2) - \frac{1}{4}\tilde\lambda(\tilde\lambda+2) = \frac{1}{4}(\lambda^2 + 2\lambda - \tilde\lambda^2 - 2\tilde\lambda)$

$= \frac{1}{4}(\lambda - \tilde\lambda)(\lambda + \tilde\lambda + 2)$

$\Rightarrow w_\lambda = w_{\tilde\lambda}$ iff $\lambda = \tilde\lambda$

e) V is simple and finite dim. $\overset{\text{thm 30}}{\Rightarrow} \exists \tilde v_0 \in V, \ \tilde H \tilde v_0 = \lambda \cdot \tilde v_0$

set $\tilde v_k = \tilde F^k \tilde v_0$, it's a basis of V.

Let $\varphi : V[\lambda] \to V$ be a linear map s.t $\varphi(v_k) = \tilde v_k$.

By thm 30, $\tilde H \tilde v_k = (\lambda - 2k)\tilde v_K$

$\tilde F \tilde v_k = \tilde v_{k+1}$, $\tilde E \tilde v_k = k(\lambda+1-k) v_{k-1}$

$\Rightarrow \varphi$ is an isomorphisme

$\rho(x) v_k = \varphi(\rho(x)\tilde v_k) \quad \forall k \quad \forall x \in sl(2,\mathbb{C})$

▨

Ex: $\lambda \in \mathbb{Z}_{\geq 0}$, $V = \{ p \in \mathbb{C}[z] \mid \deg p \leq \lambda \}$

$\partial := \frac{\partial}{\partial z}$, $E = \rho(e) = \partial$, $F = \rho(f) = \lambda \cdot z - z^2 \cdot \partial$,

$[\partial, z]p = (\partial z - z\partial)p = (zp)' - zp' = p \quad \forall p \in V$

$\Rightarrow [\partial, z] = 1$

$[E,F] = [\partial, \lambda \cdot z - z^2 \cdot \partial] = \lambda[\partial, z] - [\partial, z^2]\partial - z^2\overbrace{[\partial,\partial]}^{0}$

$= \lambda - z[\partial,z]\partial - [\partial,z]z\partial = \lambda - 2z\partial$

$\Rightarrow H = \rho(h) = \lambda - 2z\partial$

$[H,E] = [\lambda - 2z\partial, \partial] = -2[z\partial, \partial] = -2[z,\partial]\partial = 2\partial = 2E$

$[H,F] = [\lambda - 2z\partial, \lambda z - z^2 \cdot \partial] = [2(\lambda - z\partial), z(\lambda - z\partial)] =$

$= 2[\lambda - z\partial, z](\lambda - z\partial) = -2[z\partial, z](\lambda - z\partial) = -2F$

A basis of V : $v_k = z^k \quad k = 0, \dots, \lambda$

le 04.12.19

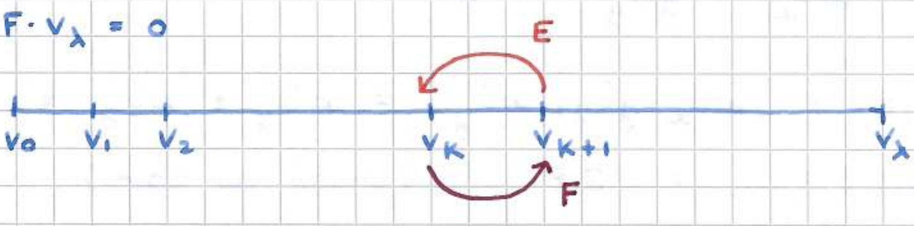
Algèbre de Lie :

$$Hv_k = (\lambda - 2z\partial)\, z^k = (\lambda - 2k)\, z^k = (\lambda - 2k)\, v_k$$

$$E.v_k = \partial z^k = k\, z^{k-1} = k\, v_{k-1}$$

$$F\cdot v_k = (\lambda z - z^2\partial)\, z^k = (\lambda - k)\, z^{k+1} = (\lambda - k)\, v_{k+1}$$

$$\Rightarrow F\cdot v_\lambda = 0$$



Ex: $\lambda \in \mathbb{Z}_{\geq 0}$ the same E, F, H as before, and $V = \mathbb{C}[z]$

a basis : $v_k = z^k$ $k = 0, 1, 2 \ldots$

$W = \text{Span}(v_0, \ldots, v_\lambda)$ is a submodule

$V \neq W \oplus$ $\Rightarrow$ V is indecomposable

Ex: $\lambda \notin \mathbb{Z}_{\geq 0}$, the same E, F, H as before and $\mathbb{C}[z] = V$

a basis $z^k = v_k$ $k = 0, 1 \ldots$

$E\cdot v_k = 0$ only for $k = 0$, $Fv_k \neq 0$ $\forall k$

Thm 32 : (Weyl)

Let A be a complex simple LA. Every finite dimentional A-module is semi-simple. (ie: $V = \bigoplus_{i=1}^{n} V_i$, V_i is simple)

(Equivalently, every finite-dim repr. of A is completly reducible)

[Without proof]

Rem: So every finite-dim $sl(2,\mathbb{C})$-module V is given by

$$V = V[\lambda_1] \oplus V[\lambda_2] \oplus \ldots$$

Ex : $N \in \mathbb{Z}_{\geq 0}$. $V_N = \{ p \in \mathbb{C}[x,y] \mid \deg p \leq N \}$

$E = x\partial_y$, $F = y\partial_x$

$[E,F] = [x\partial_y , y\partial_x] = x[\partial_y , y\partial_x] + [x, y\partial_x]\partial_y$

$= x[\partial_y , y]\partial_x + y[x,\partial_x]\partial_y$

$= x\partial_x - y\partial_y = H$

$[H,E] = [x\partial_x - y\partial_y , x\partial_y] = x\partial_y + x\partial_y = 2E$

$[H,F] = \dots = -2F$ (check as exo)

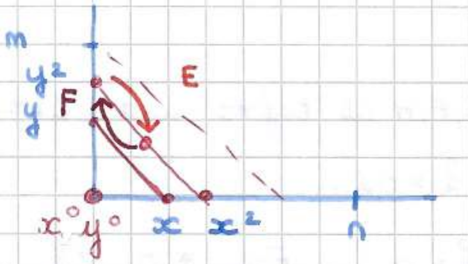
a basis of V : $v_{n,m} = x^n y^m$

a hwv : $Ev = 0$, $v_0^{(\lambda)} = x^\lambda$ $\lambda = 0,1,\dots N$

$Hv_0^{(\lambda)} = (x\partial_x - y\partial_y)v_0^{(\lambda)} = \lambda v_0^{(\lambda)}$ (λx^λ)

Set $v_K^{(\lambda)} = F^K v_0^{(\lambda)} = \lambda(\lambda-1)\dots(\lambda-K+1)\underbrace{x^{\lambda-K}y^K}_{\deg = \lambda}$

$Fv_\lambda^{(\lambda)} = y\partial_x(\dots y^\lambda) = 0$



$V_\lambda := \mathrm{Span}(v_0^{(\lambda)}, \dots, v_\lambda^{(\lambda)}) \simeq V[\lambda]$

$V_0 = \mathrm{Span}(x^0y^0)$ $V_1 = \mathrm{Span}(x,y)$

$V_2 = \mathrm{Span}(x^2, y^2, xy)$ ...

$V_\infty = \mathbb{C}[x,y] = \bigoplus_{\lambda=0}^{\infty} V[\lambda]$

le 11.12.19

Algèbre de Lie :

Tensor Product of Representations

Def: Let A and B be matrices (over $\mathbb{F}$) of arbitrary sizes.

The Kronecker product of A and B is the matrix :

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \dots \\ A_{21}B & A_{22}B & \dots \\ \vdots & & \end{pmatrix}$$

Ex : $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$

$$A \otimes B = \left(\begin{array}{cc|cc} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \\ \hline 6 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \end{array}\right)$$

Ex : $A = (1 \quad 2)$ $B = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$$A \otimes B = \begin{pmatrix} 3 & 6 \\ 4 & 8 \end{pmatrix}$$

Lemma :

a) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

b) $(A \otimes B) \cdot (C \otimes D) = A \cdot C \otimes B \cdot D$ if we can multiply the "normal" way

c) If A and B are square matrices, $tr(A \otimes B) = trA \cdot trB$

Proof :

a) $$(A \otimes B) \otimes C = \begin{pmatrix} A_{11}B & A_{12}B & \dots \\ \vdots & \vdots & \end{pmatrix} \otimes C = \begin{pmatrix} A_{11}B \otimes C & A_{12}B \otimes C \\ \vdots & \vdots \end{pmatrix}$$
$$= A \otimes (B \otimes C)$$

b) $$(A \otimes B) \cdot (C \otimes D) = \begin{pmatrix} A_{11}B & A_{12}B \dots \\ A_{21}B & A_{22}B \\ \vdots & \end{pmatrix} \begin{pmatrix} C_{11}D \dots \\ \vdots \end{pmatrix} = \begin{pmatrix} A_{11}C_{11}BD + A_{12}C_{12}BD + \dots \\ \vdots \end{pmatrix}$$
$$= \begin{pmatrix} (AC)_{11}BD + \dots \\ \vdots \end{pmatrix} = AC \otimes BD .$$

c) $A \otimes B = \begin{pmatrix} A_{11}B & & * \\ & \ddots & \\ * & & A_{nn}B \end{pmatrix}$

$$tr(A \otimes B) = \sum_{i=1}^{n} A_{ii} \, Tr(B) = Tr(A) \, Tr(B)$$

▨

Def: Let V and K be vector spaces over $\mathbb{F}$. Let $v_1 \dots v_n$ and $k_1 \dots k_m$ be basis. The tensor product of V and K is a vector space over $\mathbb{F}$, $V \otimes K$ with basis $v_i \otimes k_j$, $i=1 \dots n$, $j = 1 \dots m$, along with a bilinear map $\otimes : V \times K \longrightarrow V \otimes K$ such that $(v_i, k_j) \longmapsto v_i \otimes k_j$

Rem: By bilinearity, if $v = \sum_{i=1}^{n} \alpha_i v_i$ and $u = \sum_{j=1}^{m} \beta_j k_j$ then $v \otimes u = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \, v_i \otimes k_j$

Rem: $\dim(V \otimes U) = \dim V \cdot \dim U$.

Ex: $V = \mathbb{C}^2$, $U = \mathbb{C}^2$

$v_0 = \begin{pmatrix}1\\0\end{pmatrix}$, $v_1 = \begin{pmatrix}0\\1\end{pmatrix}$, $u_0 = \begin{pmatrix}1\\0\end{pmatrix}$, $u_1 = \begin{pmatrix}0\\1\end{pmatrix}$

$V \otimes U$, $v_0 \otimes u_0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $v_0 \otimes u_1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$, $v_1 \otimes u_0 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $v_1 \otimes u_1 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

Def: Let A be a LA over $\mathbb{F}$. Let V_1 and V_2 be finite-dimensional A-modules. Considere a linear action of A on $V_1 \otimes V_2$ such that $x \cdot (v \otimes u) = (x \cdot v) \otimes u + v \otimes (x \cdot u)$.
Then $V \otimes U$ is an A-module (Equivalently, if $\rho_1 : A \to gl(V_1)$ and $\rho_2 : A \to gl(V_2)$ are repr. of A then $\rho \equiv \rho_1 \otimes \rho_2 : A \to gl(V_1 \otimes V_2)$ such that $\rho(x)(v \otimes u) = (\rho_1(x) \cdot v) \otimes u + v \otimes (\rho_2(x) u)$ is a repr of A)

Proof: $A \times V_1 \to V_1$ and $A \times V_2 \to V_2$ are bilinear maps

$\Rightarrow A \times (V_1 \otimes V_2) \to V_1 \otimes V_2$ is bilinear.

Let $x, y \in A$ $\quad x(y(v \otimes u)) - y(x(v \otimes u)) =$

$= x((yv) \otimes u + v \otimes (yu)) - y((xv) \otimes u + v \otimes (xu))$

$= \dots = ([x,y] \cdot v) \otimes u + v \otimes ([x,y] \cdot u) = [x,y] \cdot (v \otimes u)$ ▨

$A \times (V_1 \otimes V_2) \longrightarrow V_1 \otimes V_2$ is a LA homomorphism

le 11.12.19

Algèbre de Lie :

Rem : ρ_1 and ρ_2 - two repr. of a LA A.

$$(\rho_1 \otimes \rho_2)(x)(v \otimes u) = (\rho_1(x) \cdot v) \otimes u + v \otimes (\rho_2(x) u)$$

$$\overset{KP}{=} (\rho_1(x) v) \otimes (E_{v_2} u) + (E_{v_1} v) \otimes (\rho_2(x) u)$$

$$= (\rho_1(x) \otimes E_{v_2})(v \otimes u) + (E_{v_1} \otimes \rho_2(x))(v \otimes u)$$

$$\Rightarrow (\rho_1 \otimes \rho_2)(x) = \rho_1(x) \otimes E_{v_2} + E_{v_1} \otimes \rho_2(x)$$

Ex : $A = sL(2, \mathbb{C})$, $\rho_1 = \rho_2 =$ the fund. repr

$\rho_1(e) = \rho_2(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\rho_2(f) = \rho_2(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\rho_{1,2}(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\rho = \rho_1 \otimes \rho_2$

$$\rho(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & & & \\ 1 & & 0 & \\ 1 & & & \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\rho(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} + \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix} = \begin{pmatrix} 2 & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & -2 \end{pmatrix}$$

$$C = FE + \frac{1}{4} H(H+2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 2 & & & \\ & 0 & & \\ & & 0 & \\ & & & -2 \end{pmatrix} \begin{pmatrix} 4 & & & \\ & 2 & & \\ & & 2 & \\ & & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$w = \frac{\lambda}{4}(\lambda + 2)$, $w = 2$ multiplicity ③, $w = 0$ mult. ①

$$\Rightarrow V[1] \otimes V[1] \simeq V[0] \otimes V[2]$$

$$H = \rho(h) = \begin{pmatrix} 2 & & & \\ & 0 & & \\ & & 0 & \\ & & & 2 \end{pmatrix} \Rightarrow V[1] \otimes V[1] = V_2 \oplus 2V_0 \oplus V_{-2}$$

$$\Rightarrow V[1] \otimes V[1] = (V_2 \oplus V_0 \oplus V_{-2}) \oplus V_0 \simeq V[2] \otimes V[0]$$

Def: Let V be a finite - dim sL(2-ℂ) - module.

The character of V is $\chi(V) = \sum_{\alpha} \dim(V_\alpha)\, q^\alpha$

(the sum over α such that $\dim V_\alpha \neq 0$, where $q>0$)

Exc: $V[1] = V_1 \oplus V_{-1} \rightarrow \chi(V[1]) = q + q^{-1}$

$V[0] = V_0 \longrightarrow \chi(V[0]) = 1$

Thm 34:

Let V be a finite - dimentional $sL(2,\mathbb{C})$ - module

a) $(\chi(V))_{q=1} = \dim V$

b) $\chi(V[\lambda]) = \dfrac{q^{\lambda+1} - q^{-\lambda-1}}{q - q^{-1}}$

c) $\chi(V) = tr(q^{\rho(h)})$

Proof:

a) $V \cong \bigoplus_i V[\lambda_i] = \bigoplus_i \bigoplus_{K_i=0}^{\lambda_i} V_{\lambda_i - 2K_i} = \bigoplus_\alpha V_\alpha$

$\Rightarrow \dim V = \sum_\alpha \dim V_\alpha = (\chi(V))_{q=1}$

b) $\chi(V[\lambda]) = \sum_{K=0}^{\lambda} 1 \cdot q^{\lambda-2K} = q^\lambda \left(1 + q^{-2} + \ldots + q^{-2K}\right) =$

$= q^\lambda \dfrac{1 - q^{-2\lambda-2}}{1 - q^{-\lambda}} = \dfrac{q^{\lambda+1} - q^{-\lambda-1}}{q - q^{-1}}$

c) $q^{\rho(h)} = e^{\log q \cdot \rho(h)} = \sum_{K=0}^{\infty} \dfrac{(\log q \cdot \rho(h))^K}{K!}$

if $v \in V_\alpha$ $\quad q^{\rho(h)} v = \sum_{K=0}^{\infty} \dfrac{(\log q \cdot \rho(h))^K}{K!} v = e^{\alpha \log q} \cdot v = q^\alpha v$

$\Rightarrow q^{\rho(h)}$ has $\dim V_\alpha$ eigenvalues q^α

$\Rightarrow tr(q^{\rho(h)}) = \sum$ eigenvalues $= \sum_\alpha \dim(V_\alpha)\, q^\alpha$ ▨

le 11.12.19

Algèbre de Lie :

Thm 35 : Let V_1 and V_2 be finite-dim $sl(2,\mathbb{C})$-modules.

a) $\chi(V_1 \oplus V_2) = \chi(V_1) + \chi(V_2)$

b) $\chi(V_1 \otimes V_2) = \chi(V_1) \cdot \chi(V_2)$

c) $V_1 \simeq V_2 \iff \chi(V_1) = \chi(V_2)$

d) $V_1 \otimes V_2 \simeq V_2 \otimes V_1$

e) if V_3 is another finite-dim $sl(2,\mathbb{C})$-module, then

$$(V_1 \otimes V_2) \otimes V_3 \simeq V_1 \otimes (V_2 \otimes V_3)$$

$$(V_1 \oplus V_2) \otimes V_3 \simeq (V_1 \otimes V_3) \oplus (V_2 \otimes V_3)$$

Rem : If V is a finite-dim module of $sl(2,\mathbb{C})$

$\Rightarrow$ compute $\chi(V)$ and find λ_i s.t $\chi(V) = \prod_i \chi(V[\lambda_i])$

$V \simeq \otimes_i V[\lambda_i]$

$\ulcorner$ $\chi(V) = \sum_i \chi(V[\lambda_i]) \Rightarrow V \simeq \oplus_i V[\lambda_i]$ $\lrcorner$

Ex : $V[1] \otimes V[1]$ $\quad \chi(V[1]) = q + q^{-1}$ $\quad \chi(V[2]) = q^2 + 1 + q^{-2}$

$$\chi(V[1])^2 = q^2 + 2 + q^{-2}$$

$$= q^2 + 1 + q^{-2} + 1 = \chi(V[2]) + \chi(V[0])$$

$$\Rightarrow V[1] \otimes V[1] \simeq V[2] \oplus V[0].$$

---

le 18.12.19

Recall : $\chi(V) = \sum_\alpha \dim V_\alpha \cdot q^\alpha = \mathrm{Tr}(q^{\rho(H)})$ and

$$\chi(V[\lambda]) = \sum_{k=0}^{\lambda} q^{\lambda - 2k} = \frac{q^{\lambda+1} - q^{-\lambda-1}}{q - q^{-1}}$$

Proof: a) Choose a basis of $V_1 \oplus V_2$: $\overbrace{v_1, \dots, v_n}^{\text{basis of } V_1}, \overbrace{v_{n+1}, \dots, v_m}^{\text{basis of } V_2}$

Then, $\rho_{V_1 \oplus V_2}(h) = \left(\begin{array}{c|c} \rho_{V_1}(h) & \\ \hline & \rho_{V_2}(h) \end{array}\right)$

$\ulcorner$ $q^H = e^{\ln q \cdot H} = \sum_{n \geq 0} \frac{\ln(q)^n H^n}{n!}$ $\lrcorner$

$$\Rightarrow \operatorname{Tr}\left(q^{\rho_{V_1 \oplus V_2}(h)}\right) = \operatorname{Tr}\left(\begin{array}{c|c} * & 0 \\ \hline 0 & ** \end{array}\right) = \operatorname{Tr}\left(q^{\rho_{V_1}(H)}\right) + \operatorname{Tr}\left(q^{\rho_{V_2}(H)}\right)$$

$$= \chi(V_1) + \chi(V_2)$$

b) Fix a basis of $V_1 \otimes V_2$

$$\chi(V_1 \otimes V_2) = \operatorname{Tr}\left(q^{\rho_{V_1 \otimes V_2}(h)}\right) = \operatorname{Tr}\left(q^{\overset{A}{\rho_{V_1}(h)} \otimes \overset{B}{E_2} + \overset{C}{E_1} \otimes \overset{D}{\rho_{V_2}(h)}}\right) = *$$

$$(A \otimes B) \otimes (C \otimes D) = (A \quad C) \otimes (B \quad D)$$

$$(A \otimes E) \otimes (E \otimes D) = A \otimes D = (E \otimes D) \otimes (A \otimes E)$$

$$e^{A \otimes E} = \sum_{n \geq 0} \frac{(A \otimes E)^n}{n!} = \sum_{n \geq 0} \frac{A^n \otimes E^n}{n!} = e^A \otimes E$$

$$* = \operatorname{Tr}\left(q^{\rho_{V_1}(h) \otimes E_2} \cdot q^{E_1 \otimes \rho_{V_2}(h)}\right)$$

$$= \operatorname{Tr}\left(\left(q^{\rho_{V_1}(h)} \otimes E_2\right) \cdot \left(E_1 \otimes q^{\rho_{V_2}(h)}\right)\right)$$

$$= \operatorname{Tr}\left(q^{\rho_{V_1}(h)} \otimes q^{\rho_{V_2}(h)}\right) = \operatorname{Tr}\left(q^{\rho_{V_1}(h)}\right) \operatorname{Tr}\left(q^{\rho_{V_2}(h)}\right)$$

$$= \chi(V_1) \cdot \chi(V_2)$$

c) **⇒:** $V_1 \cong V_2 \Rightarrow$ There is an invertible $\phi : V_1 \to V_2$ such that $\phi \rho_{V_1}(x) = \rho_{V_2}(x) \phi$ for all $x \in A$. Then,

$$\operatorname{Tr}\left(q^{\rho_{V_1}(h)}\right) = \operatorname{Tr}\left(q^{\phi^{-1} \rho_{V_2}(h) \phi}\right) = \operatorname{Tr}\left(\sum_{n \geq 0} \phi^{-1} (\ln q \cdot \rho_{V_2}(h))^n \phi \cdot \frac{1}{n!}\right)$$

$$= \operatorname{Tr}\left(\sum_{n \geq 0} \frac{\ln(q)^n (\rho_{V_2}(h))^n}{n!}\right) = \operatorname{Tr}\left(q^{\rho_{V_2}(h)}\right) = \chi(V_2)$$

just a finite polynomial

⇐: If $\chi(V_1) = \chi(V_2) = m_1 q^{\lambda_1} + \sum_{\lambda_i < \lambda_1} m_i q^{\lambda_i}$

$\Rightarrow V_1$ contains V_{λ_1} and every $v \in V_{\lambda_1}$ is a highest weight vector $(Ev = 0)$ (If $Ev \neq 0$, $Ev \in V_{\lambda_1 + 2}$)

$\Rightarrow V_1$ contains m_1 simple modules $V[\lambda_1]$ $(v_k = F^k v_0)$

$$\Rightarrow V_1 \cong \underbrace{V[\lambda_1] \oplus \ldots \oplus V[\lambda_1]}_{m_1 \text{ times.}} \oplus \tilde{V}_1$$

$$\Rightarrow V_2 \cong V[\lambda_1] \oplus \ldots \oplus V[\lambda_1] \oplus \tilde{V}_2$$

then by using (a), $\chi(V_1) = m_1 \chi(V[\lambda_1]) + \chi(\tilde{V}_1)$

and $\chi(V_2) = m_1 \chi(V[\lambda_1]) + \chi(\tilde{V}_2)$

$\Rightarrow \chi(\tilde{V}_2) = \chi(\tilde{V}_1) \Rightarrow$ proceed.

Algèbre de Lie:

d) $\chi(V_1 \otimes V_2) = \chi(V_1) \cdot \chi(V_2) = \chi(V_1)\chi(V_2) = \chi(V_2 \otimes V_1)$

$\Rightarrow V_1 \otimes V_2 \cong V_2 \otimes V_1$

e) $\chi((V_1 \otimes V_2) \otimes V_3) = \chi(V_1 \otimes V_2)\,\chi(V_3) = \chi(V_1)\chi(V_2)\chi(V_3)$

$= \chi(V_1)\,\chi(V_2 \otimes V_3) = \chi(V_1 \otimes (V_2 \otimes V_3))$

$\Rightarrow (V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$

Same for $(V_1 \oplus V_2) \otimes V_3 \cong V_1 \otimes V_3 \oplus V_2 \otimes V_3$

▨

Thm 36: (The Clebsh - Gordan decomposition)

Let $\lambda_1, \lambda_2 \in \overset{?}{\boxed{\mathbb{H}}}$ such that $\lambda_1 \geq \lambda_2$. Then, $V[\lambda_1] \otimes V[\lambda_2] =$

$= \bigoplus_{k=0}^{\lambda_2} V[\lambda_1 - \lambda_2 + 2k] \quad \left(= \bigoplus_{k=\lambda_1-\lambda_2}^{\lambda_1+\lambda_2} V[k]\right)$

Proof: $\sum_{k=0}^{\lambda_2} \chi(V[\lambda_1 - \lambda_2 + 2k]) = \frac{1}{q - q^{-1}} \cdot \sum_{k=0}^{\lambda_2} \left(q^{\lambda_1-\lambda_2+2k+1} - q^{-\lambda_1+\lambda_2-2k-1}\right)$

CDV $k' = \lambda_2 - k \longrightarrow$ $= \frac{1}{q - q^{-1}} \sum_{k=0}^{\lambda_2} \left(q^{\lambda_1+\lambda_2-2k+1} - q^{-\lambda_1+\lambda_2-2k-1}\right)$

$= \frac{1}{q-q^{-1}} \left(q^{\lambda_1+1} - q^{-\lambda_1-1}\right) \sum_{k=0}^{\lambda_2} q^{\lambda_2-2k} = \chi(V[\lambda_1])\,\chi(V[\lambda_2])$

thm 35
$\Rightarrow$ conclusion

▨

Ex: $\overset{3}{V[2]} \otimes (\overset{2}{V[1]} \oplus \overset{4}{V[3]}) \cong (V[2] \otimes V[1]) \oplus (V[2] \otimes V[3])$

CG
$= (V[1] \oplus V[3]) \oplus (V[1] \oplus V[3] \oplus V[5]) = 2\overset{}{V[1]} \oplus 2V[3] \oplus V[5]$

$\qquad\qquad\qquad 2 \qquad 4 \qquad 6$

Recall that $\dim(V[\lambda]) = \lambda + 1$ $\quad 2\cdot2 + 2\cdot4 + 1\cdot6 = 18$ okay !

Semi-Simple Lie Algebra:

Def: Let A be a LA and $I \subset A$ be an ideal of A.

a) I is an **abelian ideal** if $[x, y] = 0 \quad \forall x, y \in I$

b) I is a **solvable ideal** if $I^{(k)} = \{0\}$ for some $k \geq 0$

Recall: $I^{(1)} = I' = [I, I]$, $I^{(2)} = [I^{(1)}, I^{(1)}]$ etc...

Def: A LA A is **semi-simple** if $\{0\}$ is its only abelian ideal.

Rem: Every simple LA is semi-simple.

Lemma: A LA A is semi-simple iff $\{0\}$ is its only solvable ideal.

Proof: If $I \subset A$ is a solvable ideal, then $I^{(m)}$ are ideals of A (thm 40) and $I^{(k-1)}$ is an abelian LA $\Rightarrow$ A is not semi-simple.

If A has no nontrivial solvable ideals, then A has no abelian ideals (since every abelian ideal is solvable) ▨

Thm 37: (Second Cartan's criterion)

A complex LA A is semi-simple iff the Killing form of A is non-degenerate.

Proof: "only if"

Suppose that $Ker(K) \neq \{0\}$. Take $A^\perp = \{x \in A \mid tr(ad_x ad_y) = 0 \ \forall y \in A\} = Ker K$

By thm 25, $A^\perp$ is an ideal of A and K is identically zero on $A^\perp$. Then $K = 0$ on $(A^\perp)'$ $\overset{1^{st}\,C.C.}{\Rightarrow}$ $A^\perp$ is a solvable ideal of A.

Then A is not semi-simple.

▨

we do not prove the other direction but it's easy.

le 18.12.19

Algèbre de Lie :

Def : Let A_1 and A_2 be two LA. The vector space $A_1 \oplus A_2$ equipped with the Lie bracket $[\underset{\in A_1}{(x_1}, \underset{\in A_2}{y_1)}, \underset{\in A_1}{(x_2}, \underset{\in A_2}{y_2)}]_{A_1 \oplus A_2} = ([x_1, x_2]_{A_1}, [y_1, y_2]_{A_2})$ is the direct sum of LAs A_1 and A_2.

Rem : A_1 and A_2 are ideals of $A_1 \oplus A_2$

[indeed : $[(x_1, 0), (y_1, y_2)] = ([x_1, y_1], 0) \in A_1 \subset A_1 \oplus A_2$]

Lemma : If $A_1 \dots A_n$ are simple LAs, then $A = A_1 \oplus A_2 \oplus \dots \oplus A_n$ is a semi-simple LA.

Pf : Let $I \subset A$ be an abelian ideal. Take $I_i = I \cap A_i$, it's an ideal of A (thm 10)

$\Rightarrow I_i \subset A_i$ and I_i is an abelian ideal of A_i (because I abelian)

But : A_i simple $\Rightarrow I_i = \{0\}$ $\forall i \Rightarrow I = \{0\} \Rightarrow A$ semi-simple.

■

Ex : $A = \left\{ x \in sl_4(\mathbb{C}) \;\middle|\; x = \left(\begin{array}{c|c} x_1 & 0 \\ \hline 0 & x_2 \end{array}\right), \; tr(x_1) = tr(x_2) = 0 \right\}$

$\cong sl_2(\mathbb{C}) \oplus sl_2(\mathbb{C})$ is a semi-simple LA.

Thm 38 : Let A be a complex semi-simple LA. Then $A = \bigoplus_{i=1}^{n} A_i$ where A_i is a simple ideal of A (or a LA).